Painting yourself into a corner:
Graph colouring and optimization

Andrew D. King

Simon Fraser University, Burnaby, B.C.

A taste of π, December 1, 2012.

A little about me

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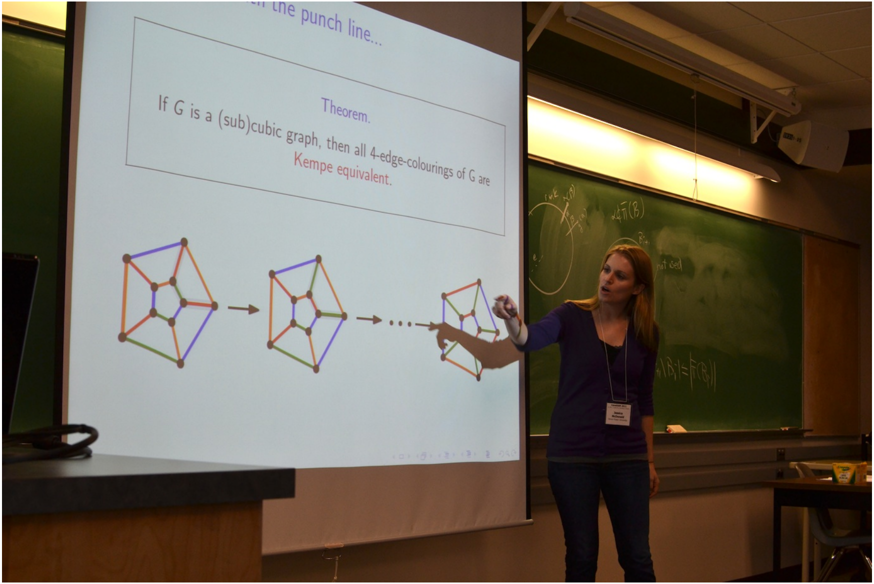
- I am a postdoctoral researcher: I finished my Ph.D. recently, and I *research* unsolved math problems.
- My research area is kind of like really complicated sudoku.
- We call it **graph theory**.

A little about me



Jessica McDonald speaking at a conference

A little about me

Daniel Kral figuring out some configurations in Montreal

A little about me

1658 *L. Esperet et al. / Advances in Mathematics 227 (2011) 1646–1664*

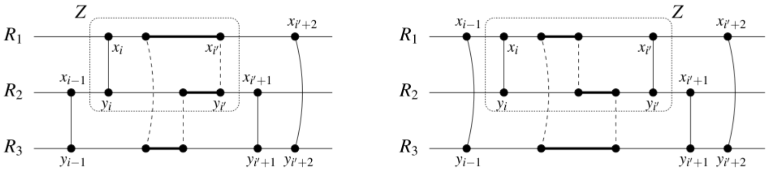
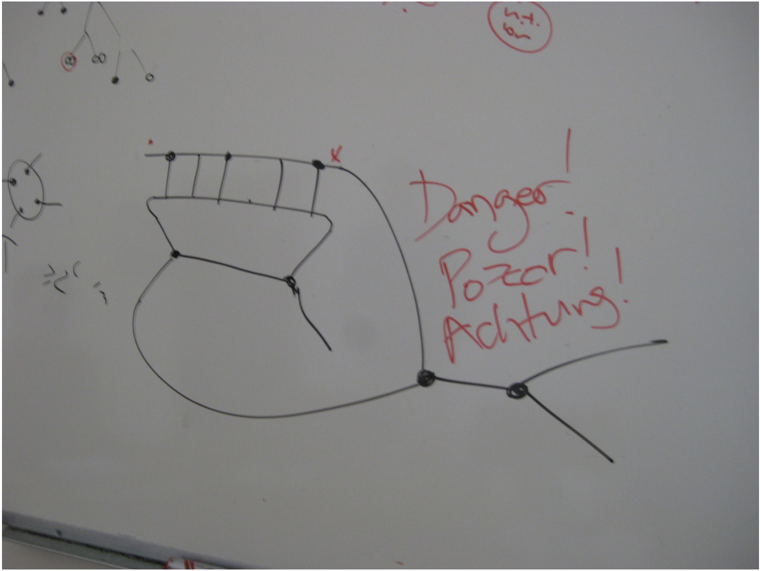


Fig. 3. Certificates for the burl X when $i' - i$ is odd (left) and when $i' - i$ is even (right). Horizontal paths are R_1, R_2 and R_3, solid edges correspond to the value 1/3 of $\mathbf{w}$, bold edges to value 2/3 and dashed edges to value 0.

Proof. Let $P' = v_{-1}v_0 \dots v_9 v_{10}$ be a path in T such that $P = v_0 \dots v_9$. Let $f_i = v_{i-1}v_i$ and let $C_i = \phi^{-1}(f_i)$, $0 \leqslant i \leqslant 10$. Let $X := \phi^{-1}(V(P))$. We assume without loss of generality that $G \mid X$ contains no cycles of length 4, as otherwise the lemma holds by Lemma 18. Let A be the set of ends of edges in C_0 outside of X, and let B be the set of ends of edges in C_{10} outside of X. Observe that E_X consists of 3 internally vertex-disjoint paths from A to B, as well as one edge in $G \mid \phi^{-1}(\{v_i\})$ for $0 \leqslant i \leqslant 9$. Let R_1, R_2 and R_3 be these three paths from A to B, and let u_j be the end of R_j in A for $j = 1, 2, 3$. For $0 \leqslant i \leqslant 9$, we have $\phi^{-1}(v_i) = \{x_i, y_i\}$ so that

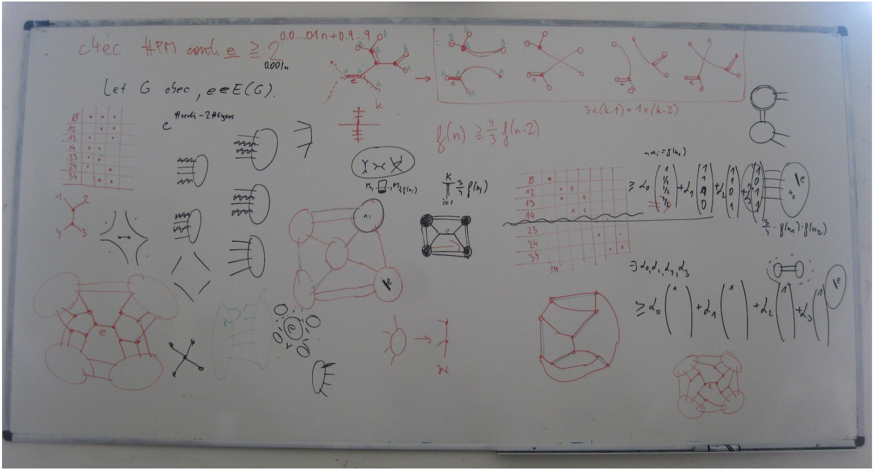
This is what the configurations turned into. We solved an old problem... older than me!

A little about me



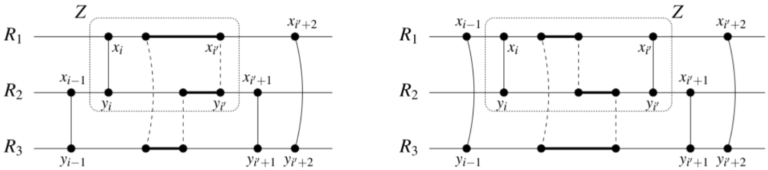
My knowledge of Czech: **pozor** and **nemazat**. Useful!

A little about me



Lots of good reasons to *pozor*.

A little about me



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A little about me

The research institute in Prague (yes, really)!

A little about me

A workshop in Barbados. Outdoor blackboards!

A little about me

Working with my Ph.D. advisor and officemate

A little about me

Pondering graph theory in Italy

A little about me

I said I study **graph theory**.
What’s a graph?

What is a graph?

Graff: graffiti

What is a graph?

Graf, Steffi

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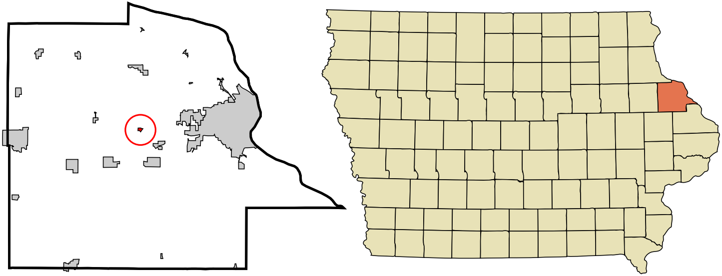

Graf, Iowa

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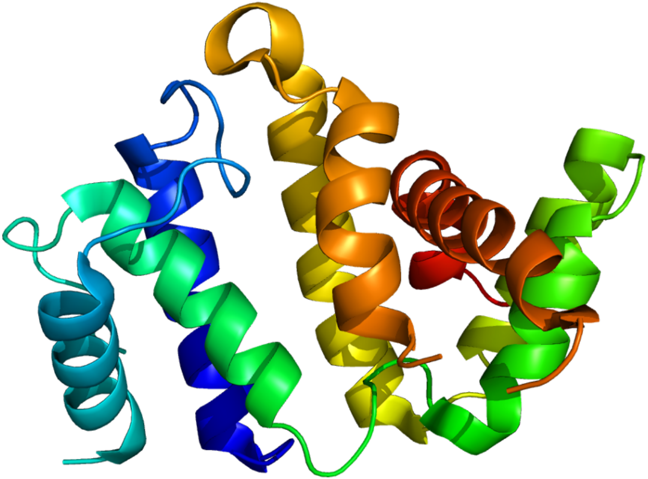
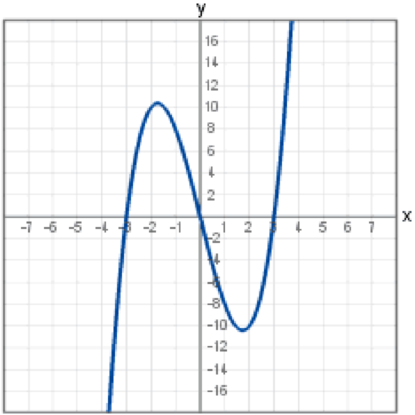


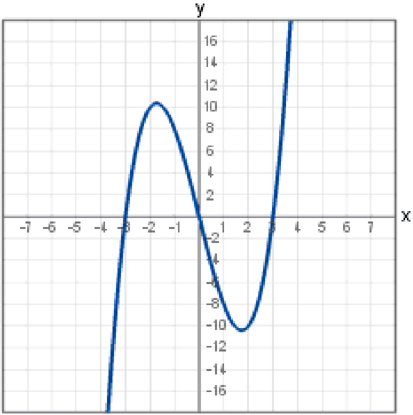
GTPase **R**egulator **A**ssociated with **F**ocal Adhesion Kinase
(GRAF)

What is a graph?



The graph of a function.

What is a graph?



The graph of a function.

Close, but *not that kind of graph.*

What is a graph?

A graph is a network of objects!

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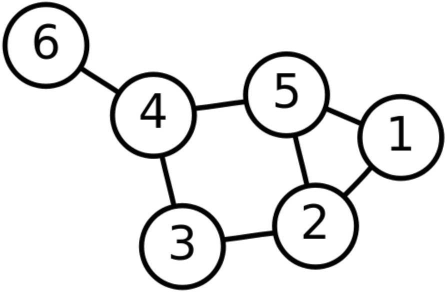

- Dots: vertices or nodes.
- Lines: edges or connections.

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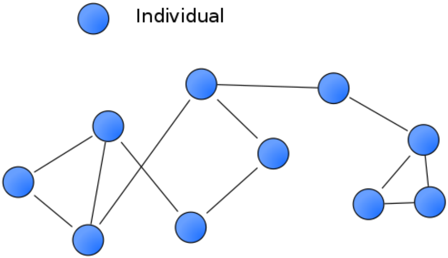

A social network, for example.
The objects are people.
The connections represent relationships.

What is a graph?

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A graph *is*

- A set of dots called vertices
- Some pairs of dots are connected.
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- Two vertices are connected or adjacent if they have an edge between them.

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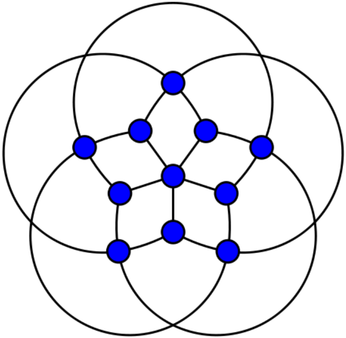
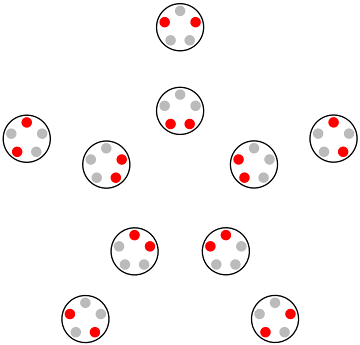


What can vertices and edges represent?

A graph can represent something abstract and mathematical or something practical... or both!



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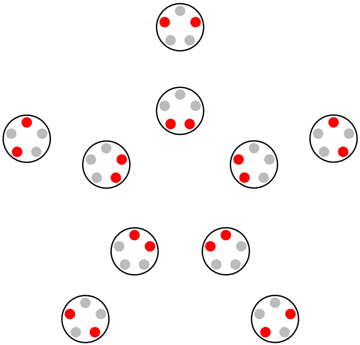

- Vertices represent combinations of two buttons.
- **Exercise:** Draw an edge between each pair of combinations that doesn't share a button.

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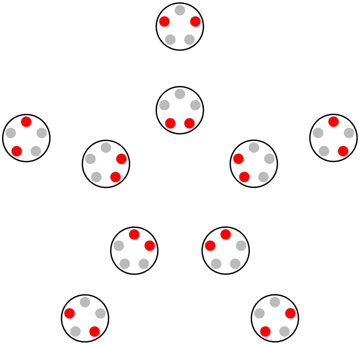

- Vertices represent tennis games between two players.
- **Exercise:** Draw an edge between each pair of games that can happen simultaneously.

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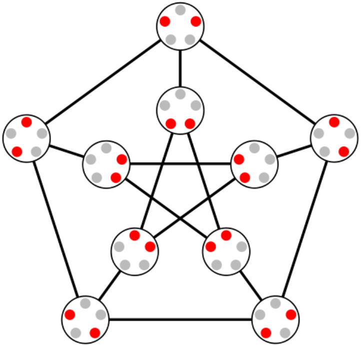

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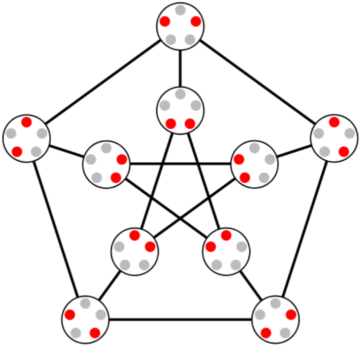


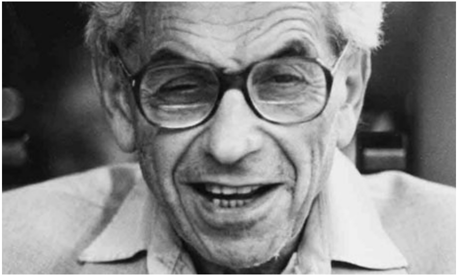
- The resulting graph is the same either way.
- Graphs provide a flexible way to model real-life problems in a mathematical setting.

Erdős Numbers

Paul Erdős, 1913–1996: A true master of graph theory.

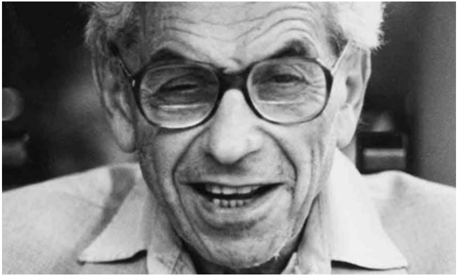
- Published over 1500 papers
- *N is a Number* – A documentary about him on Youtube.

Erdős Numbers



Erdős numbers involve interconnectedness of mathematicians!

Erdős Numbers



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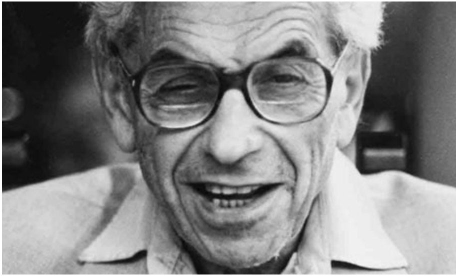

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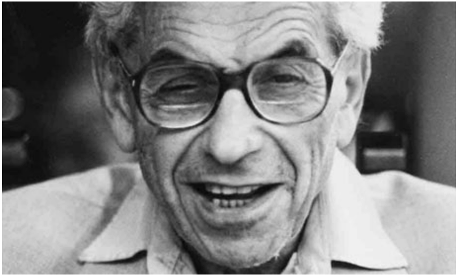

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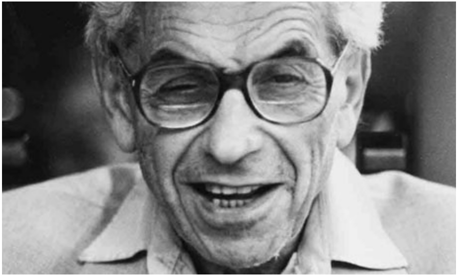

Erdős numbers involve interconnectedness of mathematicians!

- Take a graph where the vertices are mathematicians.
- Two mathematicians are connected if they published a paper together.
- How long is the shortest path from me to Erdős?
- That's my Erdős number.

Colouring a map

I am a cartographer on a budget. I can print my map in four colours, but neighbouring states should get different colours. Is this always possible?

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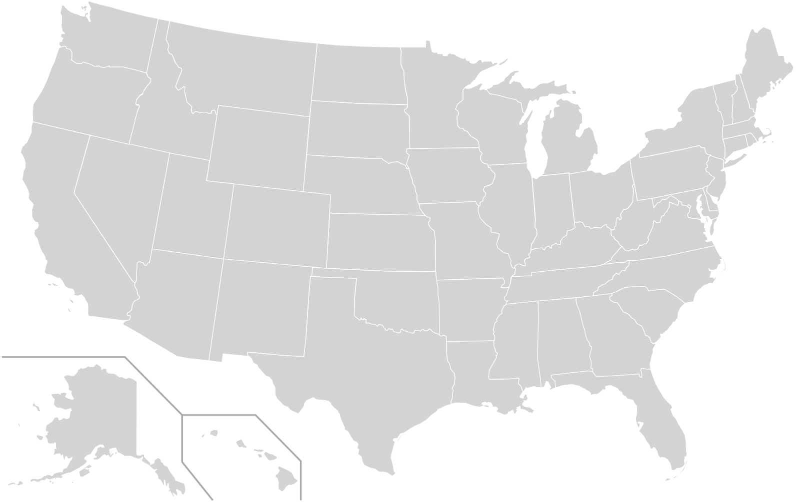

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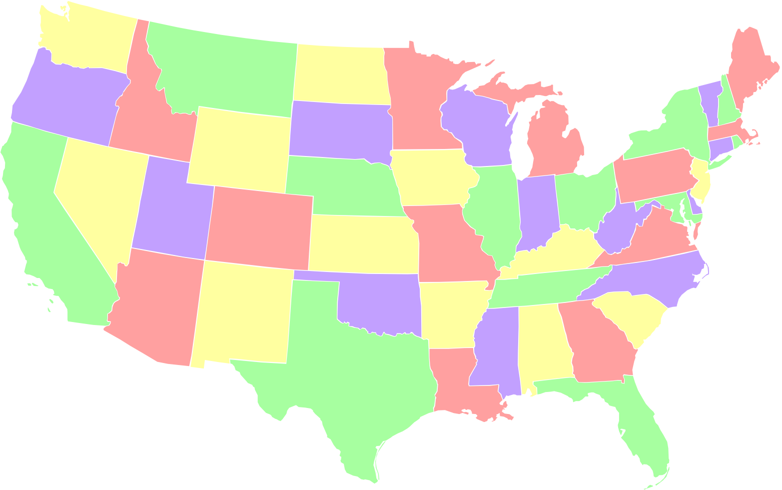

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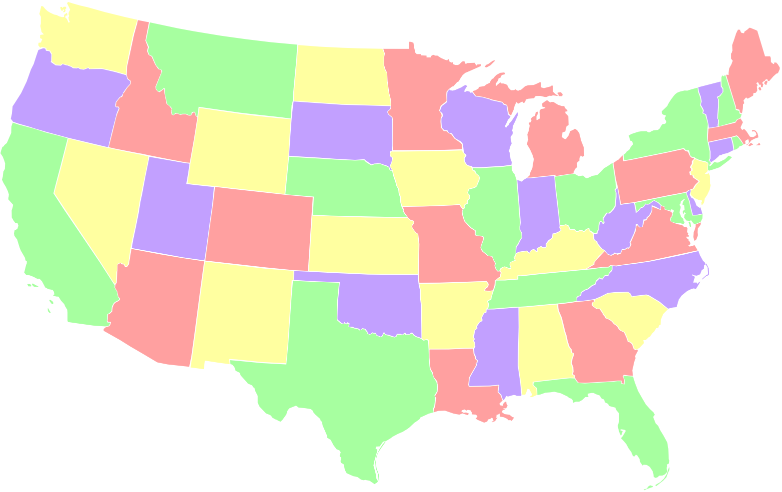
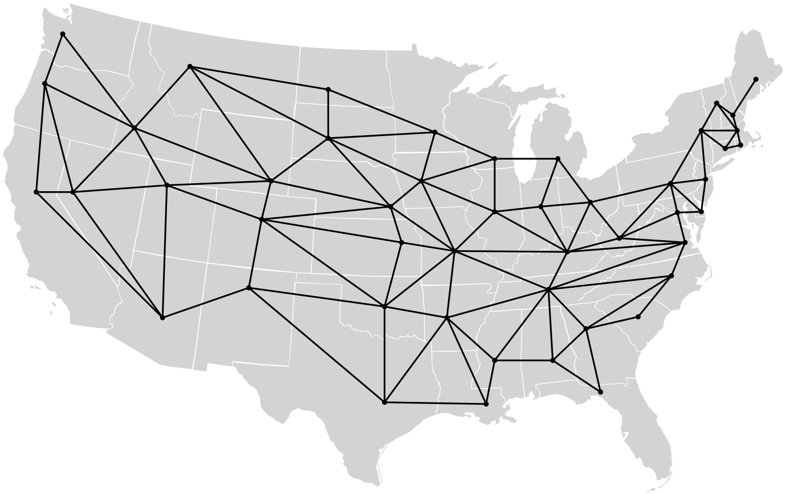
How can we view this map as a graph? Remember, neighbouring states should get different colours.



Colouring a map

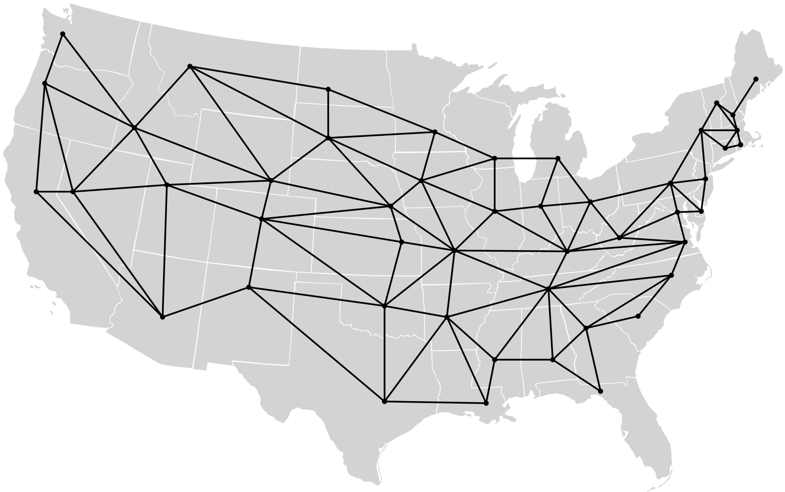
How can we view this map as a graph? Remember, neighbouring states should get different colours. Vertices are states. Neighbouring states are connected.



Colouring a map

Now we want to give the vertices colours so that no adjacent (connected) vertices get the same colour!

This is the graph colouring problem.



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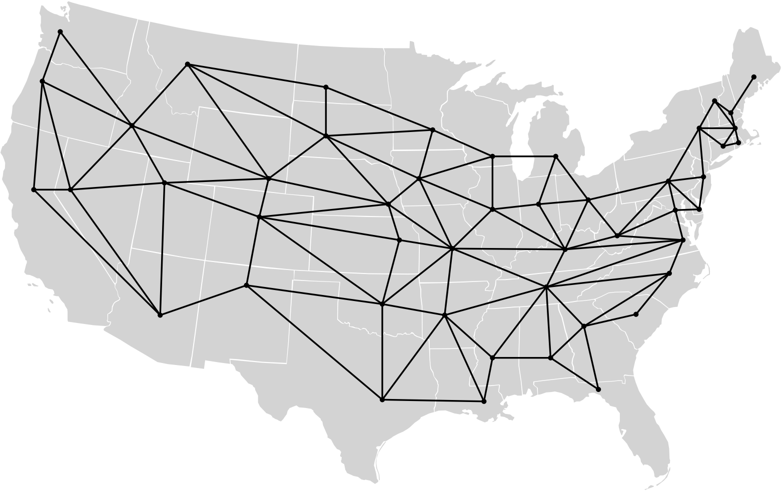
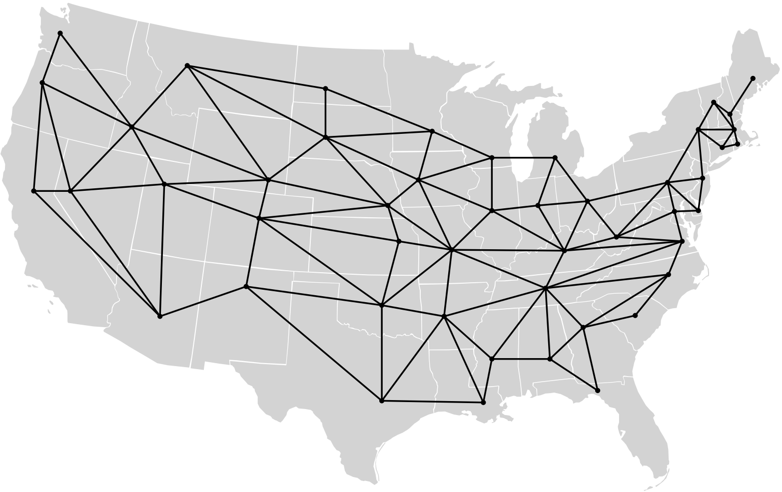
Look at Washington State.



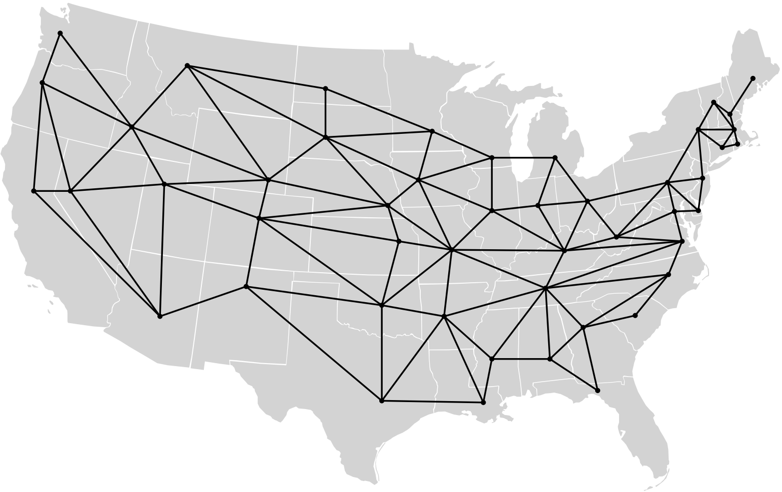
Colouring a map

Look at Washington State. Capital city?



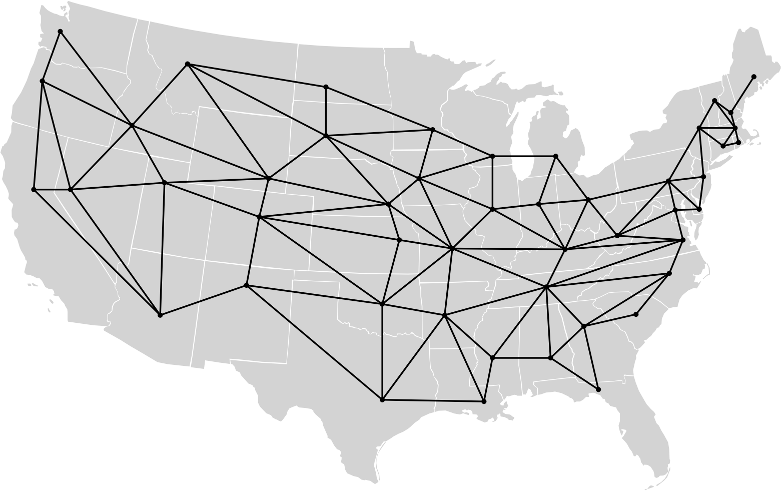
Colouring a map

Look at Washington State. Capital city? Olympia.



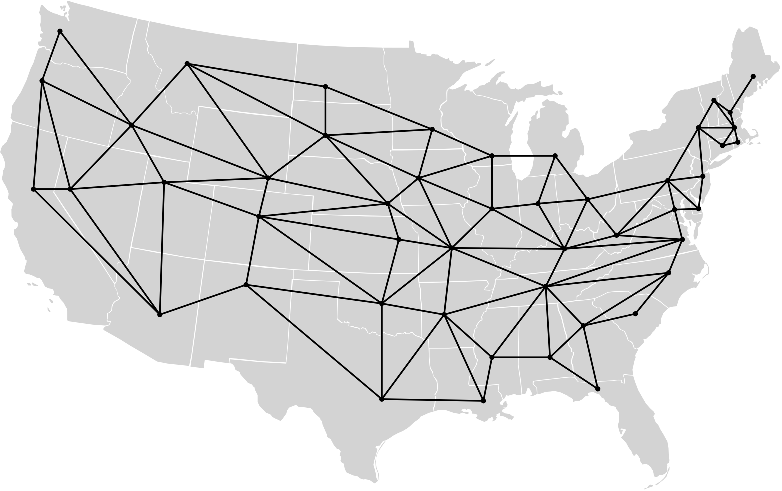
Colouring a map

Look at Washington State. Capital city? Olympia.
Washington only neighbours two states:



Colouring a map

Look at Washington State. Capital city? Olympia.
Washington only neighbours two states:
Oregon and Idaho.



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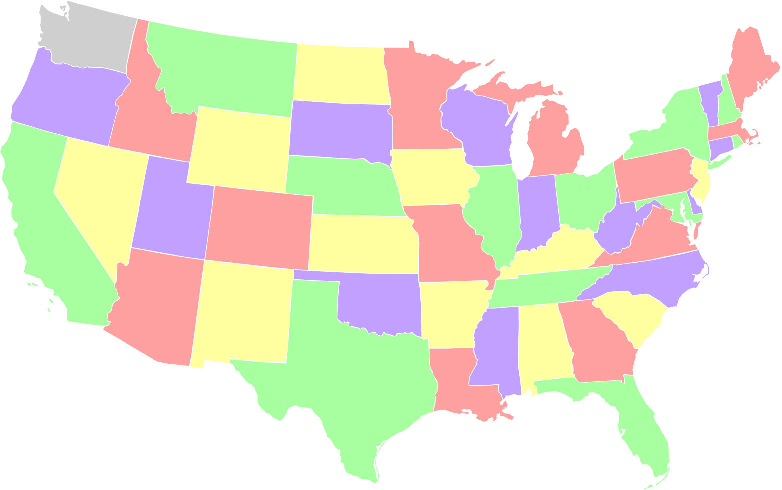
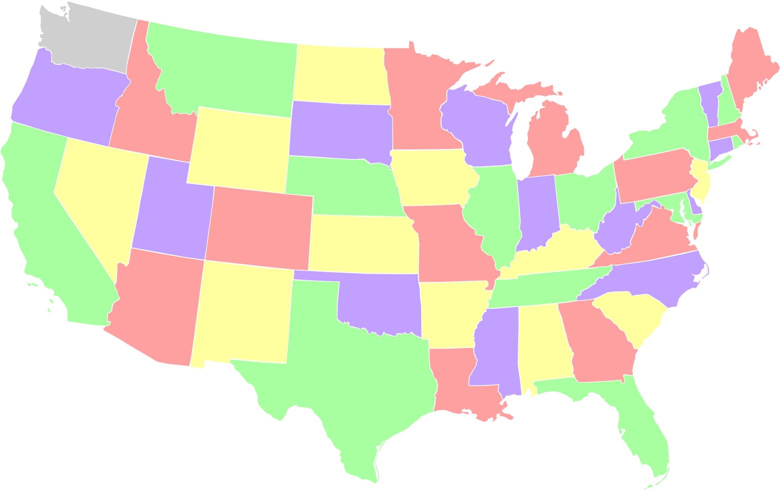


Colouring a map

If we can 4-colour everything except Washington, we can extend the colouring to Washington.

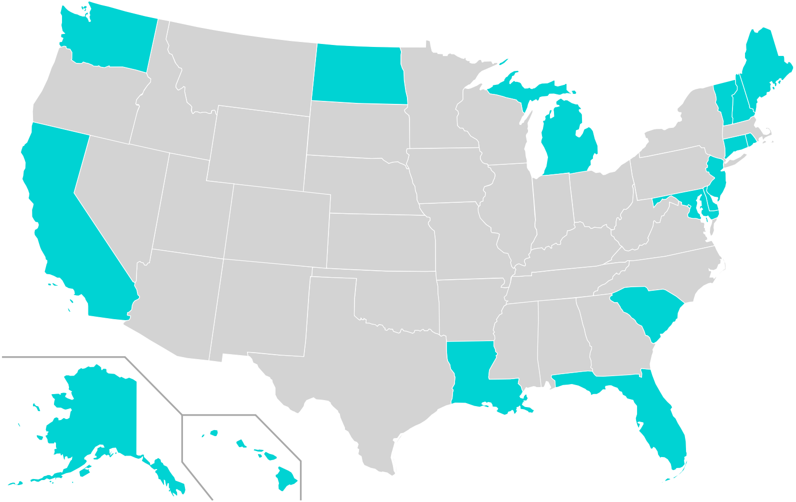
At least 4-2=2 colours are available for WA.



Colouring a map

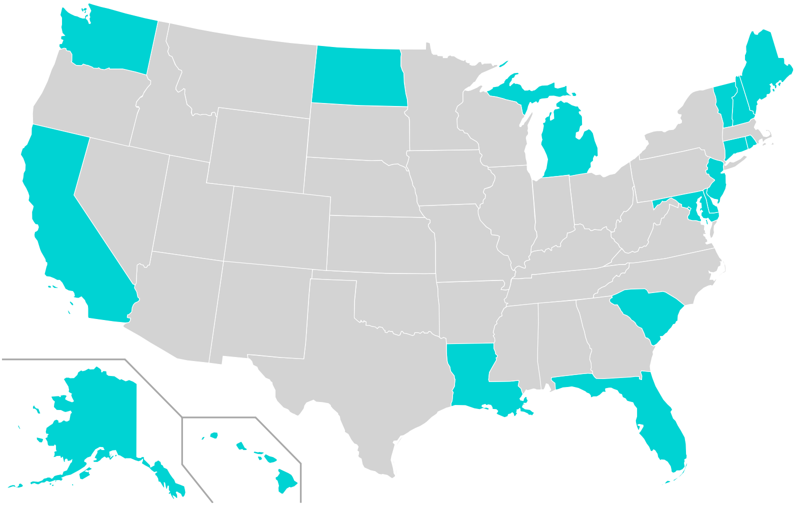
We can make the same argument for CA, FL, ME, NH, VM, RI, LA, GA, MI, NJ, CT, DE, ND, and MD.

Oh and HI and AK, obviously.



Colouring a map

So we can ignore these states, colour the rest, then extend the colouring when we are done.



Colouring a map

But we can repeat this argument. For example, Oregon only has two grey neighbours. We can remove a bunch of states this way!

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Colouring a map

Now we repeat the argument. Our map gets smaller...

Colouring a map

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And smaller...

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And smaller...

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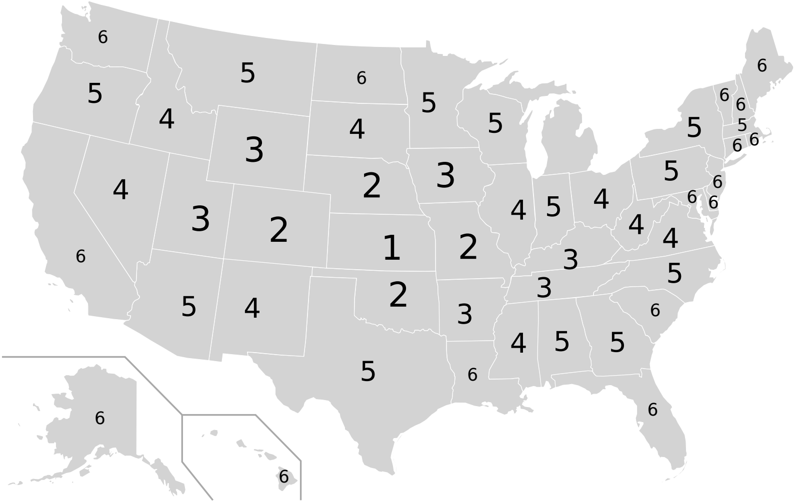
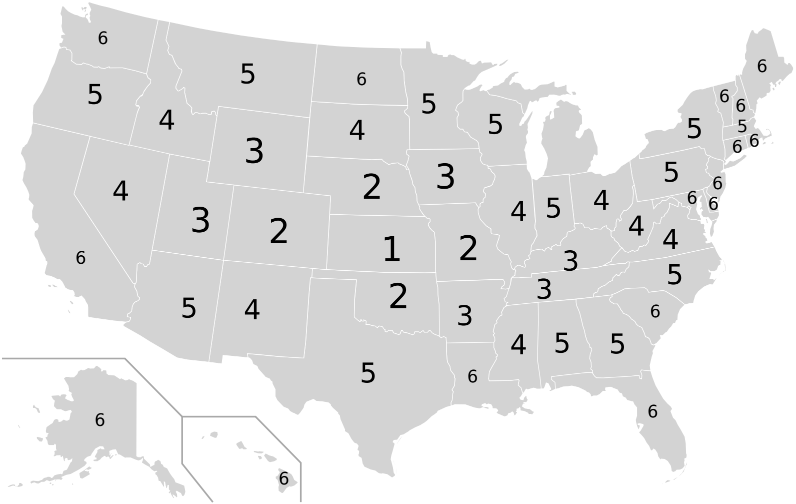
Colouring a map

Now we can colour the map in stages. First colour the state numbered 1, then extend to states numbered 2, then 3, etc.



Colouring a map

Exercise: Can you colour the states with 4 colours, stage by stage, so no two neighbouring states get the same colour?



A recursive 4-colouring algorithm

We just performed a recursive 4-colouring algorithm:

- **Recursive:** To solve the problem on our graph, we solve the same problem on a smaller graph.
- **4-colouring:** We colour the graph using 4 colours.
- **Algorithm:** A step-by-step method for solving a problem.

To 4-colour a map G:

1. Find a vertex v with at most 3 neighbours (e.g. WA).
2. Remove v and recursively 4-colour what remains.
3. Since v has at most 3 neighbours, we can extend the colouring to v.

Does this always work?

A recursive 4-colouring algorithm

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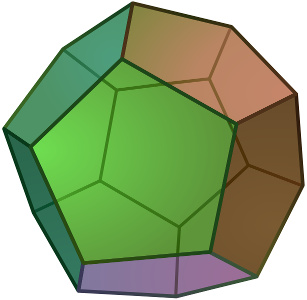

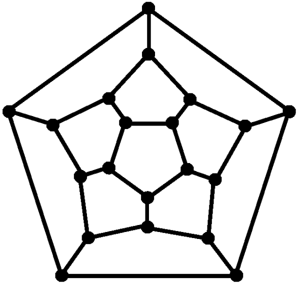

Absolutely not! Maybe v does not exist. :(

Are 4 colours always enough?

The graph associated with a map is called a planar graph, because it can be drawn in the plane (2-dimensional space) without any edges crossing each other.

Question (1852):

Is every planar graph 4-colourable?

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Why did they prove the Four Colour Theorem so many times?

4 Colour Theorem: The proof

Four Colour Theorem: 1976, 1995, 2005

Every planar graph (map) can be coloured with 4 colours.

"A proof is a proof. What kind of a proof? It’s a proof. A proof is a proof, and when you have a good proof, it’s because it’s proven." – Jean Chrétien

4 Colour Theorem: The proof

- 1976 proof was by computer.
  They proved the theorem by looking at nearly 2000 configurations.
  The computation took more than a month.
- 1995 proof was also by computer.
  They reduced the proof to about 600 configurations.
- 2005 proof was generated by a computer system that finds mathematical proofs.
  The theorem is too complicated to prove by hand!

4 Colour Theorem: The proof

Appel and Haken at work

Graph colouring

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Question: Is $P = NP$?

Translation: Can we solve NP-complete problems quickly with a "normal" computer?

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Can we give an upper bound on χ?

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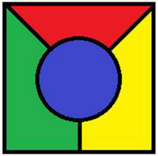
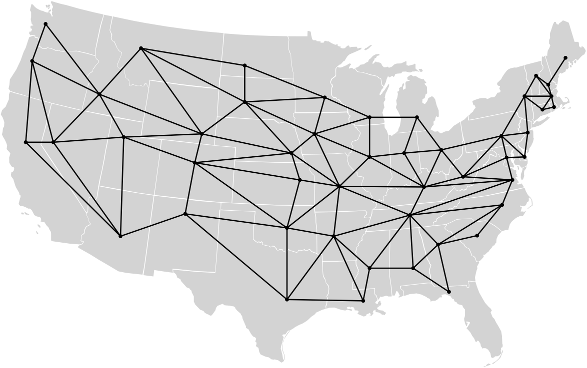
- The degree of v is the number of neighbours of v.
- The maximum degree of a graph G is the highest degree of a vertex in G.
- A clique in G is a set of vertices that are all connected to each other.



A clique of size 4 in a map

Graph colouring



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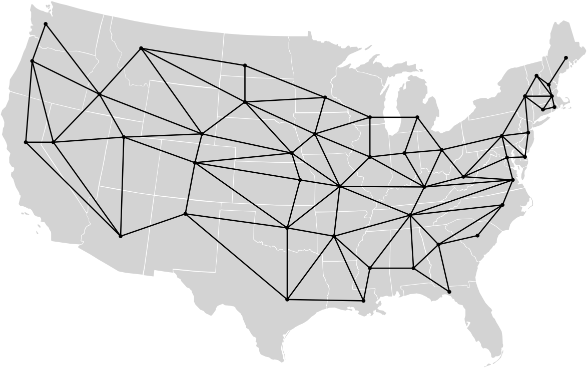
Graph colouring



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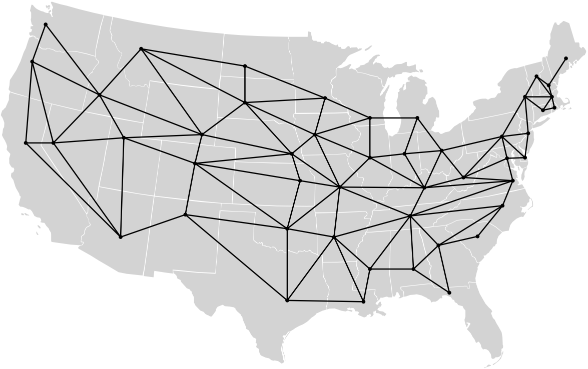

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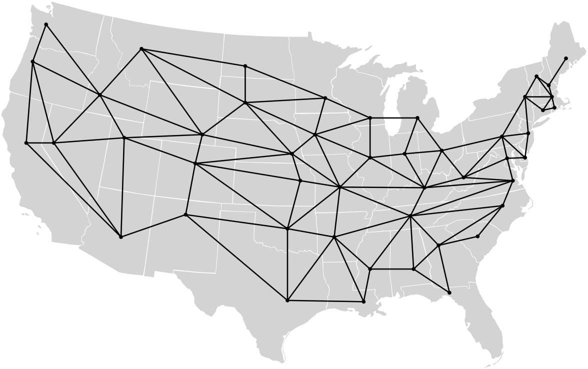

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$$3 \leq \chi(G) \leq 9$$

Graph colouring

Reed's Conjecture

The chromatic number of any graph is at most the average of the *clique number* and the *maximum degree*, plus 1.

I wrote my Ph.D. dissertation on this problem.

Sudoku: Graph colouring in disguise

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			3
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- **Game** chromatic number of G: Smallest k so that Player 1 can always win.

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- If two points are 1 apart, then they get different colours.
- Can you say why you need at least 3 colours?

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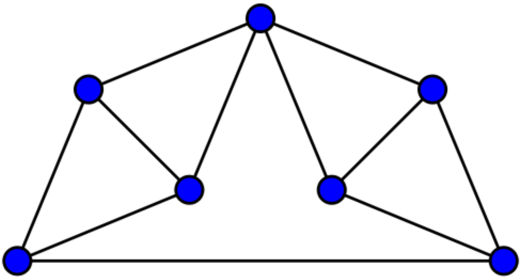

Actually you need at least 4.

This graph is called the Moser spindle.

But the bottom edge is much longer!

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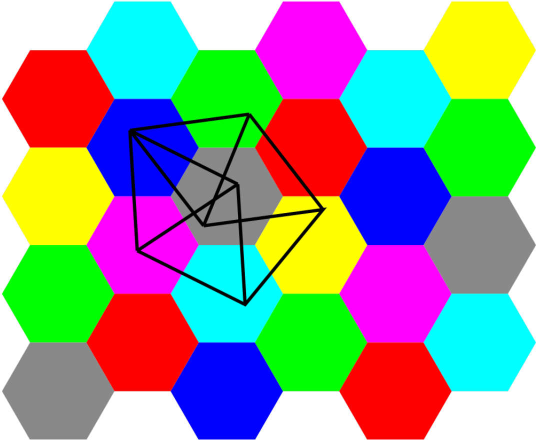

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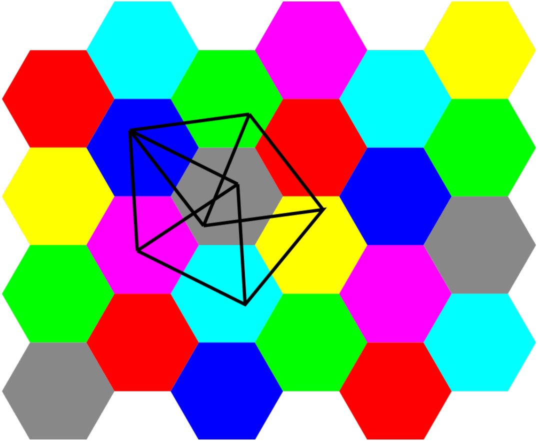

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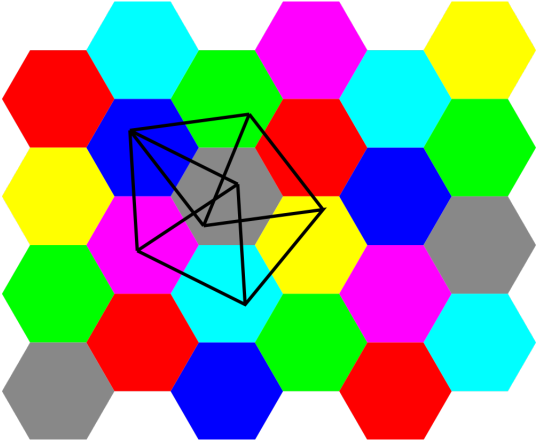

You need between 4 and 7 colours.

- Nobody knows what the answer is.
- If we change the rules a little bit, we need either 6 or 7 colours. But still, nobody knows.

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- Start with piles of size 1, then 1 or 2, and try to find the pattern.

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We want to add them without carrying.
In this world, **the only numbers are 0 and 1**, and
1+1 = 0.

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So Player 1 can win precisely if the Nim sum is not zero at the beginning of the game.

More 4-colouring maps

4-colouring a planar graph

http://www.nikoli.com/en/take_a_break/four_color_problem/

Thanks!

Thank you for your attention!

More questions? adk7@sfu.ca