The Firefighter Problem For Graphs of Maximum Degree Three

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1 Introduction

We consider a dynamic problem introduced by B. Hartnell in 1995 [4]. Let (G, r) be a connected rooted graph (where $r \in V(G)$). At time 0, a fire breaks out at r. At each subsequent time interval, the firefighter *defends* some vertex which is not on fire, and then the fire spreads to all undefended neighbours of each *burning* (i.e., on fire) vertex. Once defended, a vertex remains so for all time intervals. The process ends when the fire can no longer spread. The firefighter (optimization) problem is to determine the maximum number of vertices that can be *saved*, i.e., that are not burning when the process ends.

Papers investigating the firefighter problem have appeared in the literature. Algorithms for two- and three-dimensional grid graphs are presented in [9]. These lead to bounds on the maximum number of vertices that can be saved. NP-completeness of the firefighter problem on bipartite graphs is

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established in [10]. This paper also establishes improved bounds and exact values for two-dmensional grids, and considers the restriction of the problem to trees. The results include exponential algorithms for solving the firefighter problem on trees (one of these runs in linear time for binary trees), and a polynomial-time algorithms for a subclass of trees related to perfect graphs. It is proved in [7] that the greedy algorithm is a 2-approximation algorithm on trees, that is, the maximum number of vertices saved is never more than twice the number saved using the greedy algorithm. (It need not be the case that the number of vertices burned under a greedy strategy is at most twice the number of vertices burned under an optimum strategy.) Other aspects of the firefighter problem are studied in [2]. Related topics are examined in [1, 3, 5, 6].

Perhaps the most interesting related open problem is to determine the complexity of the firefighter problem for trees. A formal conjecture has never appeared in the literature, but it has been widely believed for some time that the problem is NP-complete for trees. We prove that this is indeed the case, and more. A sequence of transformations with increasing expressive power is used to show that the problem is NP-complete for trees of maximum degree three. This result is then used to show that the firefighter problem is NP-complete for cubic graphs. By contrast, we show that if the fire breaks out at a vertex of degree two, the problem can be solved in polynomial time for graphs of maximum degree three.

2 NP-Completeness Results

We describe a sequence of transformations involving the problems listed below (in the reverse order they are used).

3T-FIRE
Instance: A triple (T, r, k), where (T, r) is a rooted tree with $\Delta \leq 3$ and k is a positive integer.
Question: When the fire begins at r, is there a strategy such that at most k vertices burn?

3T′-FIRE
Instance: A triple (T, r, k), where (T, r) is a rooted tree such that $d(r) = 2^m + 2$ for some positive integer m, every other vertex in T has degree at

most 3, and k is a positive integer.
QUESTION: When the fire begins at r, is there a strategy such that at most k vertices burn?

3-FIRE
INSTANCE: A triple (G, r, k), where (G, r) is a rooted cubic graph, and k is a positive integer.
QUESTION: When the fire begins at r, is there a strategy such that at most k vertices burn?

The *level* (or *depth*) of a vertex x in a rooted tree (T, r) is its distance from r. The *depth of the tree* (T, r) is the maximum depth of a vertex of T. A rooted tree is called *full* if all of its leaves occur at the same level.

3FL-FIRE
INSTANCE: A full rooted tree (T, r) with maximum degree $\Delta(T) \leq 3$.
QUESTION: When the fire begins at r, is there a strategy such that no leaf burns?

3NN-SAT (Not All Equal 3-SAT without negated literals)
INSTANCE: An ordered pair (B, C) where B is a set of boolean variables and C is a set of logical clauses over B in conjunctive normal form, each containing three non-negated literals.
QUESTION: Is there a truth assignment for B such that every clause in C contains at least one true literal and at least one false literal?

3NN′-SAT
INSTANCE: An ordered pair (B, C) where B is a set of boolean variables and C is a set of logical clauses over B in conjunctive normal form, where $|B| = 2^m$ for some integer $m \geq 2$, exactly $|C|/2$ clauses in C contain only non-negated literals, and the remaining clauses are negations of these.
QUESTION: Is there a truth assignment for B such that exactly $|C|/2$ clauses in C contain exactly one true literal and the rest contain exactly two true literals?

The following straightforward lemma plays an important role in our proofs.

Lemma 1. *Suppose* (G_1, r) *is a spanning subgraph of* (G_2, r)*. Let* σ *be a* *strategy for the firefighter problem on* (G_2, r)*. Then* σ *is also a strategy for the firefighter problem on* (G_1, r)*, and will save at least as many vertices in* G_1 *as it does in* G_2*.*

Our polynomial transformations are as follows:

Theorem 2. *3NN-SAT* $\rightarrow$ *3NN'-SAT.*

Theorem 3. *3NN'-SAT* $\rightarrow$ *3T'-FIRE.*

This reduction is the central and most substantial proof of the paper. The remaining theorems in the section are proved by continuting this reduction in a straightforward manner.

Theorem 4. *3NN'-SAT* $\rightarrow$ *3T-FIRE.*

Theorem 5. *3NN'-SAT* $\rightarrow$ *3'-FIRE.*

Theorem 6. *3NN'-SAT* $\rightarrow$ *3FL-FIRE.*

The NP-completeness of 3NN-SAT (Not All Equal 3-SAT without negated literals, see [8]) thus implies that all aforementioned problems are NP-complete.

3 Polynomiality Results

Let (G, r) be a connected rooted graph with $\Delta(G) \leq 3$ and $1 \leq deg(r) \leq 2$. We consider the nontrivial case where $deg(r) = 2$. We define a function $f : V \rightarrow \mathbb{Z}^+$, computable in polynomial time, which is used to determine an optimal strategy for (G, r). The results below culminate in a polynomial time algorithm for this case.

Lemma 7. *Let* (G, r) *be a connected rooted graph, where* $1 \leq deg(r) \leq 2$*. Then there is an optimal solution to the firefighter problem in which, at each time, the vertex defended is adjacent to a burning vertex.*

Theorem 8. *Our strategy yields an optimal solution to the firefighter problem on a rooted graph* (G, r) *with* $\Delta \leq 3$ *and* $1 \leq deg(r) \leq 2$*.*

Corollary 9. *Let* (G, r) *be a connected rooted graph with* $\Delta \leq 3$ *and* $deg(r) = 2$*. The maximum number of vertices that can be saved by any strategy is* $MVS(G, r) = |V(G)| - min\{f(x) \mid x \in (V_1 \cup V_2 \cup V_c)\}$*.*

Corollary 10. *The firefighter problem is in P for graphs with maximum degree 3 in which the fire starts at a vertex of degree at most two.*

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