Bounding χ in terms of ω and Δ for quasi-line graphs

Andrew King and Bruce Reed

May 8, 2007

Abstract

A *quasi-line* graph is a graph in which the neighbourhood of any vertex can be covered by two cliques; every line graph is a quasi-line graph. Reed conjectured that for any graph G, $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$ [13]. We prove that the conjecture holds if G is a quasi-line graph, extending a result of King, Reed and Vetta, who proved the conjecture for line graphs [8], and improving the bound of $\chi(G) \leq \frac{3}{2}\omega(G)$ given by Chudnovsky and Ovetsky [2].

1 Introduction and Preliminaries

Let G be a finite simple graph. A *proper colouring* (often called just a *colouring*), is an assignment of a colour to each vertex such that no two adjacent vertices are assigned the same colour. The *chromatic number* of G, denoted $\chi(G)$, is the least number of colours required to colour the vertex set of G properly. A *clique* in G is a set of pairwise adjacent vertices; the size of the largest clique in G is called the *clique number* of G and denoted $\omega(G)$. The degree of a vertex in G is the number of vertices adjacent to it; the maximum degree over all vertices in the graph is denoted $\Delta(G)$. We will sometimes denote $\chi(G)$, $\omega(G)$, and $\Delta(G)$ by χ, ω, and Δ respectively when the graph in question is clear from the context.

For any graph G, $\chi(G) \geq \omega(G)$. To see this, note that in a proper colouring of G each vertex in a given clique must get a different colour. On the other hand, it is easy to see that G can be coloured greedily, one vertex at a time, from a set of $\Delta(G) + 1$ colours. In [13], Reed considered the problem of bounding $\chi(G)$ from above by convex combinations of the trivial lower bound $\omega(G)$ and the trivial upper bound $\Delta(G)+1$. He proved:

Theorem 1 (Reed)**.** *There is an absolute constant $a > 0$ such that for any graph G, $\chi(G) \leq \lceil a\omega(G) + (1-a)(\Delta(G)+1)\rceil$.*

Continuing in this vein, Reed conjectured that a can be as high as $1/2$. For a graph G we use $\gamma(G)$ to denote $\lceil \frac{1}{2}(\Delta(G)+1+\omega(G))\rceil$.

Conjecture 2 (Reed)**.** *For any graph G, $\chi(G) \leq \gamma(G)$.*

This conjecture seems very difficult to prove for all graphs, but it is not so bad for certain classes. There are many classes of graphs, for example perfect graphs, for which the conjecture obviously holds (a *perfect* graph is one in which any induced subgraph has equal clique number and chromatic number). Given a multigraph H, the *line graph* $L(H)$ of H is the graph with vertex set $E(H)$ in which two vertices are adjacent precisely if their corresponding edges in H share an endpoint. A graph G is a *line graph* precisely if there is a multigraph H such that $G = L(H)$. King, Reed and Vetta proved that the conjecture holds for line graphs [8].

Theorem 3 (King, Reed and Vetta)**.** *For any line graph $G = L(H)$, $\chi(G) \leq \gamma(G)$, and G can be $\gamma(G)$-coloured in $O(|V(G)|^{7/2})$ time.*

Remark: The running time of the algorithm given by King, Reed and Vetta is not explicitly stated in the paper. The algorithm appeals to an $O(|E(H)| \cdot |V(H)|)$ *edge colouring* algorithm of Nishizeki and Kashiwagi [12] in one case and removes a matching from the base graph in the other case; the matching can be found in $O(|V(H)|^{5/2})$ time and its removal reduces the maximum degree of H. The result is an algorithm that gives a $\gamma(G)$ edge colouring of H, hence a $\gamma(G)$ vertex colouring of G, in $O(|V(H)|^{7/2} + |E(H)|)$ time; it also runs in $O(|V(G)|^{7/2})$ time since $|E(H)| = |V(G)|$ and $|V(H)| = O(|E(H)|)$.

A vertex is *simplicial* if its neighbourhood induces a clique, and is *bisimplicial* if its neighbourhood can be covered by two cliques. A *quasi-line graph* is a graph in which every vertex is bisimplicial; note that every line graph is a quasi-line graph. The main result of this paper is the proof of Conjecture 2 for quasi-line graphs:

Theorem 4. *For any quasi-line graph G, $\chi(G) \leq \gamma(G)$.*

We prove in Section 6 that any quasi-line graph G can be $\gamma(G)$-coloured in polynomial time, specifically $O(n^2m^2)$ if G has n vertices and m edges.

Shannon's Theorem states that for any line graph G, $\chi(G) \leq \frac{3}{2}\omega(G)$ [15]. Chudnovsky and Ovetsky extended this result, proving that for any quasi-line graph G, $\chi(G) \leq \frac{3}{2}\omega(G)$ [2]. In a quasi-line graph G, $\Delta(G) \leq 2\omega(G) - 2$. This implies that $\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil \leq \frac{3}{2}\omega(G)$, so our result is stronger than the Chudnovsky-Ovetsky bound. The Chudnovsky-Ovetsky proof relies on a form of induction based on Seymour and Chudnovsky's structure theorem for quasi-line graphs (see [3]). To a large extent we use the same types of reductions in this paper as do Chudnovsky and Ovetsky in [2]. What makes things more difficult in this paper, generally speaking, is the maintenance of three invariants (ω, Δ, and χ) in our work rather than just two (ω and χ). Furthermore, we explicitly provide efficient algorithms which yield the desired colourings, while they do not.

We close this section with some terminology and notation. Given a graph G and a set $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S. For a vertex v and sets S and R of vertices, we denote $N(v) \cap S$ by $N_S(v)$, we denote $|N_S(v)|$ by $d_S(v)$, and we denote $max_{v \in R}\{d_S(v)\}$ by $\Delta_S(R)$. Given two (usually disjoint) vertex sets S and R, the act of *joining S to R* or *making S complete to R* consists of adding to the graph every possible edge with one endpoint in S and the other in R. For a colouring c of a graph G and a set $S \subseteq V(G)$, we use $c(S)$ to denote the set of colours used on S.

2 A Proof Sketch

In this section we state the main lemmas we need and show how they can be combined to prove Theorem 4. In order to do so, we first need to discuss some special classes of quasi-line graphs and some special types of decompositions of graphs.

A *co-bipartite* graph is a graph whose complement is bipartite, meaning precisely that its vertices can be covered by two cliques. A fundamental result of König [9] states that in a bipartite graph the size of a maximum matching and the size of a minimum vertex cover are equal – it follows that any co-bipartite graph has equal clique number and chromatic number. Since any induced subgraph of a co-bipartite graph is co-bipartite, every co-bipartite graph is perfect. Notice that a vertex v of a graph G is bisimplicial precisely if $G[N(v)]$ is co-bipartite. Results of Hopcroft and Karp provide an $O(n^{5/2})$-time algorithm for optimally colouring a co-bipartite graph and for finding a maximum clique in a co-bipartite graph [7].

A *linear interval representation* of a graph $G = (V, E)$ consists of a point on the real number line for each vertex, along with a set of intervals such that two vertices u and v of the graph are adjacent precisely if there is an interval containing both points associated with the vertices. Obviously in a representation with the fewest number of intervals, none contains another. A *linear interval graph* is a graph for which there is a linear interval representation. These graphs can be both recognized and represented in linear time [6]. Linear interval graphs can be ω-coloured in linear time by colouring the vertices greedily, moving from left to right along the real line.

A *circular interval representation* of a graph $G = (V, E)$ consists of a point on the boundary of the unit circle for each vertex, along with a set of intervals on the boundary of the unit circle such that two vertices of G are adjacent precisely if there is an interval containing both points associated with the vertices (again we assume no interval contains another). A *circular interval graph* is a graph for which there is a circular interval representation. As with linear interval graphs, circular interval graphs can be recognized and represented in linear time [6]. It is easy to prove that Conjecture 2 holds for circular interval graphs:

Lemma 5. *For any circular interval graph G, $\chi(G) \leq \gamma(G)$, and G can be $\gamma(G)$-coloured in $O(n^{3/2})$ time.*

Proof. Molloy and Reed proved in [10] that for any graph G, the fractional chromatic number $\chi_f(G)$ is at most $\frac{1}{2}(\Delta(G)+1+\omega(G))$. Niessen and Kind proved that for any circular interval graph G, $\chi(G) \leq \lceil \chi_f(G) \rceil$ [11]. Further, Shih and Hsu, improving the analysis of an algorithm developed by Teng and Tucker, obtain an $O(n^3/2)$-time algorithm for $\chi(G)$-colouring a circular interval graph [16]. The lemma follows. □

Henceforth, when we are given a circular (resp. linear) interval graph on n vertices we will assume that the vertices are labeled $v_1, v_2, \ldots, v_n$ in clockwise (resp. left-to-right, i.e. ascending) order. When we say that we take a stable set greedily from left to right, we take the leftmost vertex and move from left to right, adding a vertex if it has no neighbour in the stable set.

We say that a pair of disjoint cliques (A, B) form a *homogeneous pair of cliques* if $|A|, |B| \geq 2$, $|A| + |B| < n$, and for any vertex v not in $A \cup B$, v sees all or none of $|A|$ and all or none of $|B|$. If $G[A \cup B]$ contains an induced C_4, then (A, B) is a *nontrivial homogeneous pair of cliques*, otherwise it is a *trivial homogeneous pair of cliques*. Homogeneous pairs were first defined by Chvátal and Sbihi in a paper on bull-free perfect graphs [4]; this work was later built upon by Maffray and Reed to obtain a characterization of claw-free perfect graphs [14].

A *2-join* in a graph consists of four disjoint (possibly empty) sets of vertices X_i, Y_i, $i \in \{1, 2\}$ and a partitioning of the graph's vertex set into V_1 and V_2 such that $X_i \cup Y_i \subseteq V_i$ for $i \in \{1, 2\}$ and an edge between V_1 and V_2 exists precisely if the endpoints are both in $X_1 \cup X_2$ or they are both in $Y_1 \cup Y_2$. We denote the 2-join by $((X_1, Y_1), (X_2, Y_2))$; the partitioning (V_1, V_2) is implicit in a connected graph and we denote the induced subgraphs $G[V_1]$ and $G[V_2]$ by G_1 and G_2 respectively. Cornuéjols and Cunningham introduced 2-joins as a special case of 2-amalgams [5].

We modify the notion of a 2-join, defining an *interval 2-join* in the same way as a 2-join except we allow $X_1 \cap Y_1$ and $X_2 \cap Y_2$ to be nonempty and we insist first that X_i and Y_i are cliques for $i \in \{1, 2\}$ and second that there be a linear interval representation of G_2 with X_2 and Y_2 at the extreme left and right of the representation, respectively. We say that an interval 2-join is *trivial* if $V_2 = X_2 = Y_2$, and we say that a nontrivial interval 2-join is *canonical* if $X_2 \cap Y_2$ is empty. Given a nontrivial interval 2-join $((X_1, Y_1), (X_2, Y_2))$ where $C = X_2 \cap Y_2$, observe that $((X_1 \cup C, Y_1 \cup C), (X_2 \setminus C, Y_2 \setminus C))$ is a canonical interval 2-join.

With the preceding machinery in hand, we are now in a position to state the main lemmas. The first is a result that follows from a structure theorem of Chudnovsky and Seymour [3], which we will discuss in Section 5.

Lemma 6. *Every quasi-line graph which is not a line graph or a circular interval graph either contains a nontrivial homogeneous pair of cliques or admits a nontrivial interval 2-join.*

Complementing this are two results on the structure of a minimum counterexample to the main theorem.

Lemma 7. *Let G be a counterexample to Theorem 4 on a minimum number of vertices and subject to that containing a minimum number of edges. Then G contains no nontrivial homogeneous pair of cliques.*

Lemma 8. *Let G be a counterexample to Theorem 4 on a minimum number of vertices and subject to that containing a minimum number of edges. Then G does not admit a nontrivial interval 2-join.*

Given these three lemmas, proving the main theorem comes down to proving Conjecture 2 for line graphs (already done in [8]) and circular interval graphs (already done in this section).

3 Dealing With Homogeneous Pairs

In this section we prove Lemma 7, which states that a minimum counterexample to Theorem 4 contains no nontrivial homogeneous pair of cliques. We actually prove an algorithmic variant of this result.

Lemma 9. *Let G be a quasi-line graph on n vertices containing a nontrivial homogeneous pair of cliques (A, B). In $O(n^{5/2})$ time we can find a proper subgraph H of G such that $\chi(H) = \chi(G)$, and such that given a k-colouring of H we can find a k-colouring of G in $O(n^{5/2})$ time.*

Before proving this lemma, let us simply point out that it implies Lemma 7 because if G is a counterexample to Theorem 4 then H is a smaller counterexample.

Proof of Lemma 9. Let X be a maximum clique in $G[A \cup B]$; we can find X in $O(n^{5/2})$ time as mentioned in Section 2. Denote $A \setminus X$ and $B \setminus X$ by A' and B' respectively. We construct H from G by removing all edges between A' and B and all edges between B' and A. H is a proper subgraph of G because $G[A \cup B]$ contains an induced C_4 while $H[A \cup B]$ does not. Note that both $G[A \cup B]$ and $H[A \cup B]$ are cobipartite and have maximum clique size $|X|$, so we can $|X|$-colour them in $O(|A \cup B|^{5/2})$ time.

We must show that H is quasi-line. Suppose a vertex v is not bisimplicial in H and let (S, T) be a partitioning of $N_G(v)$ into two cliques. If v has a neighbour $w \in S \setminus (A \cup B)$ that sees A but not B, then $B \subseteq T$ and thus $S \cup A$ and $T \setminus A$ are two cliques covering $N_H(v)$ in H. By symmetry we can assume that if no such w exists then all of $N_H(v) \setminus (A \cup B)$ sees $A \cup B$, therefore $(S \cup A) \setminus B$ and $(T \cup B) \setminus A$ are two cliques covering $N_H(v)$ in H. Therefore H is quasi-line.

Let c_H be a proper colouring of H using $k \geq \chi(H)$ colours. Since (A, B) is a homogeneous pair, to construct a k-colouring of G, it is enough to find a colouring of $G[A \cup B]$ that uses the same set of colours as c_H on A and on B. We can do this in $O(n^{5/2})$ time because the number of colours which appear on both A and B in the colouring of H is at most the maximum size of a matching in $\bar{H}$, which is the same as the size of a maximum matching in $\bar{G}$, i.e. $|(A \cup B) - X|$. □

As we will later be pressed to detect a nontrivial homogeneous pair of cliques in polynomial time, we close the section by describing how to do so. Since such a pair contains a 4-hole, we proceed by checking, for every edge contained in a 4-hole, whether or not there is a homogeneous pair of cliques such that this edge is in one of the cliques.

We consider any edge a_1a_2 appearing in an induced C_4; whether or not an edge is in a C_4 can be determined in $O(m)$ time. We then iteratively grow cliques A_i and B_i such that if there is a homogeneous pair of cliques (A, B) with $a_1, a_2 \in A$, then $A_i \subseteq A$ and $B_i \subseteq B$. Observe that if b_1b_2 is an edge and $G[\{a_1, a_2, b_1, b_2\}]$ is a C_4 then b_1 and b_2 must be in B and so if such an (A, B) exists it is a nontrivial homogeneous pair of cliques. Let $A_0 = \{a_1, a_2\}$ and let $B_0 = \emptyset$. For $t = 1, 2, \ldots n-2$ we do the following.

1. Search for a vertex v not in $A_{t-1} \cup B_{t-1}$ that sees some but not all of A_{t-1} (resp. B_{t-1}) – it must be in B (resp. A), so let $B_t = B_{t-1} \cup \{v\}$ (resp. $A_t = A_{t-1} \cup \{v\}$) and increment t. If there is no such v then (A_{t-1}, B_{t-1}) form a nontrivial homogeneous pair of cliques; return this fact and terminate.

2. If A_t and B_t are not both cliques or $t = n-2$, terminate. A and B do not exist.

When building (A, B) we only add a vertex to the homogeneous pair if it cannot be outside the pair, hence we never face the possibility of putting an unnecessary vertex in $A \cup B$. It follows that if our method fails there is no homogeneous pair of cliques containing A_0. The method is clearly polytime: we can construct (A_t, B_t) from (A_{t-1}, B_{t-1}) in $O(m)$ time, and there are $O(m)$ possible edges to check, so the total running time is at most $O(nm^2)$. However, by taking a little care when growing our cliques, we can determine whether or not there is a vertex that needs to be put into A or B without examining any edge or non-edge more than once throughout the whole run for a given A_0. Hence we can find a minimal nontrivial homogeneous pair of cliques, or determine that there is none, in $O(n^2m)$ time.

4 Dealing With Interval 2-joins

In this section we prove Lemma 8, addressing the case in which G contains an interval 2-join. We need to colour the linear interval graph G_2 whilst ensuring our colouring can be combined with our colouring of G_1.

At this point some motivation for our method is in order. Suppose a graph F is the result of a clique sum of two graphs F_1 and F_2. That is, F is reached from F_1 and F_2 by taking a clique of the same size in each graph and identifying them. Suppose further that we can easily colour F_1 and F_2 with c_1 and c_2 colours respectively. Then we can $(\max\{c_1, c_2\})$-colour F by permuting the colour classes in one of the colourings and taking the clique sum of the two coloured graphs; this clique sum will provide a proper colouring of F. This idea is relevant to our situation in two ways. First, if some X_i or Y_i is empty then our 2-join amounts to a clique cutset and G_2 is a linear interval graph. In this case we can $\gamma(G_1)$-colour G_1 in polytime and easily extend the colouring to a $\gamma(G)$-colouring of G. Second, we use a generalization of this idea to prove Lemma 10, dealing with some cases by pasting together colourings on sets that are not necessarily cliques.

Before stating the lemma we give some notation. Given G yielding a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ let H_2 denote $G[V_2 \cup X_1 \cup Y_1]$. Define $\omega'(H_2)$ as the size of the largest clique in H_2 not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$ and define $\gamma'(H_2)$ as $\lceil \frac{1}{2}(\Delta_G(V_2 \cup X_1 \cup Y_1) + 1 + \omega'(H_2)) \rceil$, noting that this invariant is at most $\gamma(G)$. Observe that we can compute $\omega'(H_2)$ in $O(m)$ time since a clique defining this invariant will have size $|X_1 \cup X_2|$, $|Y_1 \cup Y_2|$, $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$, or $\omega(G_2)$.

Recall that we consider X_2 to be at the extreme left of G_2 and Y_2 to be on the extreme right. Lemma 8 follows easily from the following lemma in the same way that Lemma 7 follows from Lemma 9 in the previous section since $\max\{\gamma(G_1), \gamma'(H_2)\} \leq \gamma(G)$.

Lemma 10. *Let G be a quasi-line graph on n vertices and suppose G admits a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l-colouring of G_1 for any $l \geq \gamma'(H_2)$ we can find a proper colouring of G using l colours in $O(nm)$ time.*

Proof. Denote $\Delta_G(V_2 \cup X_1 \cup Y_1)$ by D, denote $\omega'(H_2)$ by W, and denote $D - W$ by S. Note that $l \geq W + \frac{S}{2}$.

We proceed by induction on l, observing that the case $l = 1$ is trivial. We begin by modifying the colouring so that the number k of colours used in both X_1 and Y_1 in the l-colouring of G_1 is maximal. That is, if a vertex $v \in X_1$ gets a colour that is not seen in Y_1, then every colour appearing in Y_1 appears in $N(v)$. This can be done in $O(n^2)$ time. If l exceeds $\gamma'(H_2)$ we can just remove a colour class in G_1 and apply induction on what remains. Thus we can assume that $l = \gamma'(H_2)$ and so if we apply induction we must remove a stable set whose removal lowers both l and $\gamma'(H_2)$.

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the colouring of G_1 to an l-colouring of G in one step. In other cases we remove a colour class in G_1 together with vertices in G_2 such that everything we remove is a stable set, and when we remove it we reduce both W and D (and hence $\gamma'(H_2)$); after doing this we apply induction on l. Notice that if $X_1 \cap Y_1 \neq \emptyset$ and there are edges between X_2 and Y_2 we may have a large clique in H_2 which contains some but not all of X_1 and some but not all of Y_1; this is not necessarily obvious but we deal with it in every applicable case.

In some cases we colour G_2 mod W. This can be done in $O(m)$ time, and the removal of any colour class of such a colouring lowers the degree of any vertex in G_2

with at least $W-1$ neighbours in G_2; we use this implicitly along with the fact that $W-1 \leq D$ to show that D drops when we remove such a colour class along with other vertices provided that we take a vertex in $X_1 \cup X_2$ and a vertex in $Y_1 \cup Y_2$.

Case 1. $X_1 = Y_1$:

H_2 is a circular interval graph and X_1 is a clique cutset. We can $\gamma(H_2)$-colour H_2 in $O(n^{3/2})$ time using Lemma 5. By permuting the colour classes we can ensure that this colouring agrees with the colouring of G_1. If $X_1 = Y_1$ then $\omega(H_2) = W$, so $\gamma(H_2) \leq \gamma'(H_2) \leq l$ and we are done.

Case 2. No colour appears in both X_1 and Y_1, i.e. $k = 0$:

By maximality of k, $\Delta_G(X_1) \geq |X_1|+|Y_1|+|X_2|$ and $\Delta_G(Y_1) \geq |X_1|+|Y_1|+|Y_2|$; also note that X_1 and Y_1 are disjoint. Now, $|X_1| + |Y_1| = W + b$ for some b. If $b > s$ then $l > D$ and hence we can extend the l-colouring of G_1 to an l-colouring of G greedily.

Now assume $b \leq 0$. Construct the circular interval graph H_3 from H_2 by adding every possible edge between X_1 and Y_1, and $\gamma(H_3)$-colour H_3 in using Lemma 5. The bound on $|X_1| + |Y_1|$ guarantees that $\omega(H_3) = W$. Since every vertex in $X_1 \cup Y_1$ gets a unique colour in this colouring, we can easily paste the colouring of H_3 to the colouring of G_1 to reach a colouring of G. By our bound on $\Delta_G(X_1)$ we have $\Delta(H_3) \leq D$, so $\gamma(H_3) \leq l$. Therefore the colouring we get is an l-colouring of G.

In the remaining case, $0 < b \leq S$. There are nonadjacent $x \in X_1$ and $y \in Y_1$. In G_1, every colour appears in the closed neighbourhood of either x or y by maximality of k. Furthermore, the at least $W + b$ colours on $X_1 \cup Y_1$ must appear in both closed neighbourhoods. It follows that one of these vertices has at least $W-1+\frac{b}{2}+\frac{S}{4}$ coloured neighbours, at least $\max\{0, \frac{S-2b}{4}\}$ of which have colours not appearing in X_1 or Y_1; assume that it is x (the same argument works if it is y). This implies that $|X_2| \leq \frac{3S-2b}{4}$. We take a W-colouring of G_2 and then recolour $\min\{|X_2|, \frac{S}{2}\}$ vertices of X_2 using new colours. This yields a colouring of G_2 with at most $\max\{0, \frac{S-2b}{4}\}$ colours appearing in both X_2 and Y_2; our aim is to paste this colouring onto the colouring of G_1 with no conflicts. To this end we first match these colours appearing in both X_2 and Y_2 with some of the $\max\{0, \frac{S-2b}{4}\}$ colours not appearing in $X_1 \cup Y_1$, then we match the remaining colours in X_2 with some of those appearing Y_1; we know that $|Y_1| > |X_2|$ since $|X_1|+|X_2| \leq W$ and $|X_1|+|Y_1| > W$. Similarly we know that $|X_1| > |Y_2|$, so we can now match the remaining colours appearing in Y_2

with some of those in X_1. We may still have unmatched colours in G_2 that appear in neither X_2 nor Y_2, but we can match these with unmatched colours in G_1 to reach a proper l-colouring of G since there are at most $W + \frac{S}{2} \leq l$ total colours in G_2.

Case 3. $k > 0$ and $|X_2| + |Y_2| < W$.

We W-colour G_2 mod W; some colour does not appear in $X_2 \cup Y_2$. Furthermore we can insist that this colour is used in G_2 unless $G_2 = X_2 \cup Y_2$. Remove the union of this colour class (which may be empty) and some colour class in G_1 appearing in both X_1 and Y_1 (take one hitting their intersection if $X_1 \cap Y_1$ is nonempty). Apply induction to the remaining graph, observing that when we remove the vertices we lower both D and W.

Case 4. $k > 0$ and $|X_2| + |Y_2| > W$, and there are no edges between X_2 and Y_2.

Since both $|X_1|+|X_2|$ and $|Y_1|+|Y_2|$ are each at most W, $|X_1 \cup Y_1|+|X_2|+|Y_2| \leq 2W$. Therefore $|X_1 \cup Y_1| < W$ and there is some colour class c not seen on this set. Colour G_2 mod W; some colour appears in both X_2 and Y_2. Remove this colour class along with c and apply induction to the remaining graph. Again our reduced graph has lower D and W.

Case 5. $k > 0$ and $|X_2| + |Y_2| > W$, and there is some edge between X_2 and Y_2.

Consider a largest clique C with vertices in both X_2 and Y_2. Pick one endpoint of this clique and the vertex beside the other endpoint (outside C) in the linear interval representation of G_2; these two vertices are nonadjacent and are both in $X_2 \cup Y_2$. Remove these two vertices along with some colour class in G_1 not hitting $X_1 \cup Y_1$ (whose existence is guaranteed as in the previous case), then apply induction to the remaining graph. Removing the vertices lowers D, and we take a vertex from every maximum clique in G_2 and every maximum clique in $G_2[X_2 \cup Y_2]$, ensuring that W also drops.

Case 6. $k > 0$ and $|X_2| + |Y_2| = W$.

Colour G_2 mod W. If there is a colour class hitting neither X_2 nor Y_2 then remove it along with a colour class in G_1 hitting both X_1 and Y_1, taking one that hits their intersection if their intersection is not empty; this lowers both D and W. Hence we can assume that there is no colour class hitting neither X_2 nor Y_2 and consequently none hitting both X_2 and Y_2.

If there is a colour class in G_1 hitting X_1 but not Y_1, we remove it along with a colour class in G_2 hitting Y_2 but not X_2; the symmetric argument applies if there is a colour class in G_1 hitting Y_1 but not X_1, hence we can assume that X_1 and Y_1 get the same colours and therefore have equal size. If there is a vertex $x \in X_1 \setminus Y_1$ with degree less than $W - 1 + \frac{S}{2}$ in G_1, there is a colour not seen in its closed neighbourhood and we can recolour it; this allows us to remove its new colour class in G_1 along with a colour class in G_2 appearing in Y_2 but not X_2. When we remove these vertices we lower D, and we can ensure that we lower W by taking the colour class in G_2 containing either the rightmost vertex of X_2 or the leftmost vertex of Y_2.

Now we can assume the existence of a vertex $x \in X_1 \setminus Y_1$ with at least $W-1+\frac{S}{2}$ neighbours in G_1. Since it has at most $D = W - 1 + S$ neighbours in total, we know that $|X_2| \leq \frac{S}{2}$. To complete the colouring of G we first colour X_2 with colours not appearing in $X_1 \cup Y_1$. We then colour the remainder of G_2 greedily from right to left, starting with Y_2. When we colour a vertex v its coloured neighbours outside X_2 form a clique of size at most $W - 1$ since v sees all of these vertices, hence v has at most $W + \frac{S}{2} - 1$ coloured neighbours and we can complete the colouring.

These cases cover every possibility, so we need only prove that the colouring can be found in $O(nm)$ time. If k has been maximized and we apply induction, k will stay maximized: every vertex in $X_1 \cup Y_1$ will have every remaining colour in its closed neighbourhood except possibly if we recolour a vertex in Case 6. In this case the overlap in what remains is $k - 1$, which is the most possible since we remove a vertex from X_1 or Y_1, each of which has size k. Hence need only maximize k once. We can determine which case applies in $O(m)$ time, and it is not hard to confirm that whenever we extend the colouring in one step our work can be done in $O(nm)$ time. When we apply induction, i.e. in steps 3, 4, 5, and possibly 6, all our work can be done in $O(m)$ time. Since $l < n$ it follows that the entire l-colouring can be completed in $O(nm)$ time. □

5 The structure of quasi-line graphs

In this section we will motivate and state Chudnovsky and Seymour's structure theorem for quasi-line graphs, then use it to prove Lemma 6, thus completing the proof of the main theorem as decribed in Section 2.

Observe that if we are given a multigraph H, to find its line graph $L(H)$ we can take $|V(H)|$ disjoint cliques corresponding to the vertices of H, where the clique corresponding to a vertex v has $d(v)$ vertices, each labeled by an edge having v as an endpoint. Then, for each edge e of H in turn, we replace the two vertices labeled with e by a single vertex adjacent to the union of their neighbourhoods.

To generalize this idea, we now define a method of graph composition that can be used to generate all quasi-line graphs containing no nontrivial homogeneous pair of cliques. Given a claw-free graph S with two simplicial vertices a and b, we say that (S, a, b) is a *strip*. If S is a linear interval graph on vertices $v_1, \ldots, v_n$ in order, then (S, v_1, v_n) is a *linear interval strip*.

We compose n strips $(S'_1, a'_1, b'_1), \ldots, (S'_n, a'_n, b'_n)$ as follows. Let S_0 be a graph on $2n$ vertices in which each connected component is a clique, and whose vertices are in n disjoint pairs, namely $(a_1, b_1), \ldots, (a_n, b_n)$. For $i = 1, \ldots, n$, let S_i be the graph obtained by taking the disjoint union of S_{i-1} and S'_i, making $N_{S_{i-1}}(a_i)$ complete to $N_{S'_i}(a'_i)$ and $N_{S_{i-1}}(b_i)$ complete to $N_{S'_i}(b'_i)$, and finally removing the vertices a_i, b_i, a'_i, and b'_i.

Observe that if every strip is a path on three vertices then the composition operation is equivalent to the construction of a line graph; the middle vertex of each strip corresponds to an edge in the base graph. Further, if every strip is a clique C along with ends a and b both with neighbourhood C, their composition is a line graph; the vertices in C correspond to $|C|$ edges between the same two vertices in the base graph. Finally, note that the composition operation is commutative in the sense that if we swap (a_i, b_i) with (a_{i+1}, b_{i+1}) in S_0 and (S'_i, a'_i, b'_i) with $(S'_{i+1}, a'_{i+1}, b'_{i+1})$ for some $1 \leq i < n$, the resulting graph S_{i+1} does not change.

Chudnovsky and Seymour have a structure theorem for quasi-line graphs [3], to which we append a corollary that is used implicitly in [2]. Note that fuzzy circular interval graphs and fuzzy linear interval strips are generalizations of circular interval graphs and linear interval strips, respectively, and are defined in [3]. They are not needed in this paper.

Theorem 11 (Chudnovsky and Seymour)**.** *Let G be a connected quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

Corollary 12. *Let G be a connected quasi-line graph with no nontrivial homogeneous pair. Then G is either a circular interval graph or a composition of linear interval strips.*

We are now equipped to prove the main structural lemma of this paper, Lemma 6, which is essentially a corollary of Corollary 12.

Proof of Lemma 6. If a quasi-line graph G containing no nontrivial homogeneous pair of cliques is neither a line graph nor a circular interval graph, it is the composition of at least two linear interval strips $(S'_1, a'_1, b'_1), \ldots, (S'_k, a'_k, b'_k)$. One of these strips must not be a clique C joined to ends a and b, and by commutativity of the composition operation we can assume that it is (S'_k, a'_k, b'_k). Observe that the composition of this final strip corresponds to an interval 2-join in G, and it cannot be trivial otherwise the strip would have the precise structure we have forbidden. □

6 Algorithmic Considerations

We can colour circular interval graphs with γ colours in $O(n^{3/2})$ time by Lemma 5 and we can colour line graphs with γ colours in $O(n^{7/2})$ time by Theorem 3. We now show that for any quasi-line graph which is neither a circular interval graph nor a line graph, we can find in $O(n^2 m)$ time a quasi-line graph H with fewer edges than G such that given a $\gamma(H)$-colouring of H we can construct a $\gamma(G)$-colouring of G in $O(n^{5/2} + nm)$ time. Combining these results yields an $O(n^2m^2 + n^{5/2}m)$-time algorithm to γ-colour quasi-line graphs.

There are three types of reductions that we use: removing simplicial vertices, reducing on a nontrivial homogeneous pair of cliques and reducing on an interval 2-join. If G has a simplicial vertex we can find and remove it in $O(nm)$ time, colour the remaining graph, then give the simplicial vertex any colour not appearing in its neighbourhood. If G has a nontrivial homogeneous pair of cliques we apply Lemma 9, removing at least one edge to reach H. If neither of these cases applies then G contains a nontrivial interval 2-join, in which we apply Lemma 10.

In light of Lemmas 6, 9 and 10, along with the algorithm for finding a nontrivial homogeneous pair of cliques given in Section 3, our desired result is implied by the following lemma, which we spend the rest of the section proving. For a vertex v, the *closed neighbourhood of* v, denoted $\bar{N}(v)$, is $N(v) \cup \{v\}$. Recall that if G contains no canonical interval 2-join it contains no nontrivial interval 2-join.

Lemma 13. *Let G be a quasi-line graph containing no nontrivial homogeneous pair of cliques and no simplicial vertex. In time $O(n^2m)$ we can find a canonical interval 2-join in G or determine that none exists.*

Proof. If a canonical interval 2-join exists (call it $((X_1, Y_1), (X_2, Y_2))$ as usual), there are nonadjacent vertices x and y in G such that $G[V_2]$ has a linear interval representation with x and y at the extreme left and right. We proceed by guessing x and y, then checking to see if they yield a desired join.

Suppose $((X_1, Y_1), (X_2, Y_2))$ is a canonical interval 2-join. Since x is not simplicial both X_1 and $N(x) \setminus (X_1 \cup X_2)$ are nonempty. Thus $\bar{N}(x)$ has exactly two maximal cliques, namely $X_1 \cup X_2$ and $\bar{N}(x) \setminus X_1$. We can find one maximal clique greedily in linear time, and having generated one of them, C, we can find the other by generating a maximal clique in $\bar{N}(x)$ containing some arbitrarily chosen element of $\bar{N}(x) \setminus C$. Thus, X_2 is the intersection of these two maximal cliques and there are two choices for X_1. In the same vein we know Y_2 and have two choices for Y_1. For each of these four possible choices of (X_1, Y_1) we first check if we indeed have a 2-join. We check that it is an interval 2-join by adding to G_2 vertices x' and y' with neighbourhoods X_2 and Y_2 respectively along with a vertex z with neighbourhood $\{x', y'\}$, then checking to see if the result is a circular interval graph. All of this can be done in $O(m)$ time, so checking every possible x and y takes $O(n^2m)$ time. □

7 Conclusion

Conjecture 2, if true, will likely be very difficult to prove in general. Proving the conjecture for claw-free graphs, however, may be substantially easier thanks to the structural characterization found in the work of Chudnovsky and Seymour [3]. In terms of promising classes for which the conjecture might be proved with relative ease, one should also consider the class of even-hole-free graphs – whereas in a quasi-line graph every vertex is bisimplicial, it was recently proved by Addario-Berry, Chudnovsky, Havet, Reed and Seymour that every even-hole-free graph has a bisimplicial vertex [1]. Their main theorem actually states something slightly stronger, but the idea of constructing any even-hole-free graph by iteratively adding a bisimplicial vertex may be very helpful for proving the conjecture.

References

[1] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed, and P. Seymour. Bisimplicial vertices in even-hole-free graphs, 2006. Manuscript.

[2] M. Chudnovsky and A. Ovetsky. Coloring quasi-line graphs. *Journal of Graph Theory*, 54:41–50, 2007.

[3] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B. S. Webb, editor, *London Mathematical Society Lecture Note Series: Surveys in Combinatorics, 2005.* Cambridge University Press, 2005.

[4] V. Chvátal and N. Sbihi. Bull-free Berge graphs are perfect. *Graphs and Combinatorics*, 3:127–139, 1987.

[5] G. Cornuéjols and W. H. Cunningham. Compositions for perfect graphs. *Discrete Mathematics*, 55:245–254, 1985.

[6] X. Deng, P. Hell, and J. Huang. Linear-time representation algorithms for proper circular-arc graphs and proper interval graphs. *SIAM Journal on Computing*, 25:390–403, 1996.

[7] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2:225–231, 1973.

[8] A. King, B. Reed, and A. Vetta. An upper bound for the chromatic number of line graphs. *Discrete Mathematics*. To appear.

[9] D. König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Mathematische Annalen*, 77:453–465, 1916.

[10] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer-Verlag, Berlin, 2000.

[11] T. Niessen and J. Kind. The round-up property of the fractional chromatic number for proper circular arc graphs. *Journal of Graph Theory*, 33:256–267, 2000.

[12] T. Nishizeki and K. Kashiwagi. On the 1.1 edge-coloring of multigraphs. *SIAM Journal on Discrete Mathematics*, 3:391–410, 1990.

[13] B. Reed. ω, Δ, and χ. *Journal of Graph Theory*, 27:177–212, 1998.

[14] B. Reed and F. Maffray. A description of claw-free perfect graphs. *Journal of Combinatorial Theory Series B*, 75:134–156, 1999.

[15] C. E. Shannon. A theorem on coloring the lines of a network. *Journal of Mathematics and Physics*, 28:148–151, 1949.

[16] W.-K. Shih and W.-L. Hsu. An $O(n^{3/2})$ algorithm to color proper circular arcs. *Discrete Applied Mathematics*, 25:321–323, 1989.

[17] A. Teng and A. Tucker. An $O(qn)$ algorithm to q-color a proper family of circular arcs. *Discrete Mathematics*, 55:233–243, 1985.