# Claw-free graphs and two conjectures on omega, Delta, and chi

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# Abstract


This thesis concerns the relationship between four graph invariants: $\omega$, $\chi_f$, $\chi$, and $\Delta$. These are the clique number, the fractional chromatic number, the chromatic number, and the maximum degree, respectively[1]. Trivially $\omega \leq \chi_f \leq \chi \leq \Delta + 1$. We seek to improve the upper bound on $\chi$. We are motivated by a conjecture of Reed, which essentially states that $\chi$ is at most the average of its trivial upper and lower bounds:

**Conjecture.** *For any graph,* $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.

We call this the Main Conjecture, and propose a Local Strengthening based on the neighbourhood of a single vertex:

**Conjecture.** *For any graph* $G$, $\chi \leq \max_{v \in V(G)} \lceil \frac{1}{2}(d(v) + 1 + \omega(G[\bar{N}(v)])) \rceil$.

We begin by showing that much of the early evidence supporting the Main Conjecture also supports the Local Strengthening. In particular, the variant of the Local Strengthening obtained by replacing $\chi$ by $\chi_f$ holds, as does the Local Strengthening when the stability number is two.

Guided by the first of these results we look towards line graphs, for which $\chi_f$ and $\chi$ agree asymptotically. We prove the Main Conjecture for line graphs, then we seek to generalize this result.

To do this we use recent results of Chudnovsky and Seymour, who characterized the structure of all claw-free graphs. We refine their results by introducing a graph reduction on certain types of homogeneous pairs of cliques that preserves the chromatic number. Thus we need only consider the problem of colouring *skeletal* claw-free graphs, which cannot be reduced. The structure of skeletal claw-free graphs is simpler than that of general claw-free graphs.

We generalize two results from line graphs to the class of quasi-line graphs. Namely, that the Main Conjecture holds, and that $\chi_f$ and $\chi$ agree asymptotically. We then consider all claw-free graphs. We prove the Main Conjecture for all claw-free graphs and we prove the Local Strengthening for claw-free graphs with a three-colourable complement. Our proofs yield polynomial-time colouring algorithms that achieve the conjectured bounds.


[1] We use standard graph theory notation, which can be found in the glossary.

# Résumé


Cette thèse a pour sujet la relation entre quatre invariants de graphes : $\omega$, $\chi_f$, $\chi$, et $\Delta$. Il s'agit respectivement du nombre de clique, du nombre chromatique fractionnaire, du nombre chromatique, et du degré maximum. Ces paramètres vérifient trivialement l'encadrement suivant : $\omega \leq \chi_f \leq \chi \leq \Delta + 1$, dans lequel on cherche à améliorer la borne supérieure sur $\chi$. Une des principales motivations pour ce travail est une conjecture de Reed, qui dit essentiellement que $\chi$ est au plus la moyenne de ses bornes inférieures et supérieures triviales.

**Conjecture.** *Pour tout graph,* $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.

On appelle cet énoncé la Conjecture Principale, et on propose un Renforcement Local basé sur le voisinage de chaque sommet.

**Conjecture.** *Pour tout graphe* $G$, $\chi \leq \max_{v \in V(G)} \lceil \frac{1}{2}(d(v) + 1 + \omega(G[\bar{N}(v)])) \rceil$.

On commence par montrer que la plupart des arguments en faveur de la Conjecture Principale incitent également à croire que le Renforcement Local est vrai. En particulier, la borne donnée par le Renforcement Local vaut pour $\chi_f$ et le Renforcement Local peut être montré lorsque le nombre de stabilité vaut deux.

Guidé par ces premiers pas, on s'intéresse aux graphes adjoints, pour lesquels $\chi_f$ et $\chi$ sont asymptotiquement équivalents. On montre la Conjecture Principale dans le cas des graphes adjoints et on cherche ensuite à généraliser ce résultat.

Pour cela on utilise des résultats récents de Chudnovsky et Seymour, qui ont caractérisé la structure les graphes sans griffes. On affine ces résultats en introduisant la notion de graphes squelettes. Dans les problèmes auxquels on s'intéresse, on peut facilement se ramener au cas des graphes squelettes, et la structure des graphes squelettes sans griffes est plus simple que celle des graphes sans griffes en général.

On étend deux résultats des graphes adjoints aux graphes quasi-adjoints : on montre que la Conjecture Principale est vérifiée pour ces graphes et que leurs nombres chromatique et chromatique fractionnaire sont asymptotiquement équivalents. On considère ensuite l'ensemble des graphes sans griffes, qui sont construits de deux manières différentes par deux opérations de composition, selon que $\chi(\overline{G}) \leq 3$ ou pas. On prouve la Conjecture Principale pour tous les graphes sans griffes et le Renforcement Local pour les graphes sans griffes dont le complémentaire est 3-colorable. Les preuves de ces résultats donnent des algorithmes de coloration en temps polynomial.

# Acknowledgements

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Upon my arrival at McGill I was thrown into a hive of scum and villainy known most often as MC109. Students, couches, bikes, and years came and went, and there I made some lifelong friends and collaborators: Louigi, Nicolas, Sean, Conor, Zhentao, and all the others. And especially my brother and third floor transplant Jamie, who pointed me towards McGill in the first place. Who's in town, and who's turn is it to bring coffee? It was too hot in the winter, too cold in the summer, and always too much fun. But as the funding bodies say, all good things must come to an end. I couldn't have fallen in with a better crowd and I'm very grateful for it.

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Thanks to my parents, for being patient and loving to their itinerant sons. Thanks to Geneviève for her patience with my French and many other things. And thanks to all my friends, whom I will not presume to enumerate, delimit or truncate, but you know who you are. You have made the years go too fast.

# Contents

# Part I
# $\omega$, $\Delta$, and $\chi$

It is well-known and easy to prove that for any graph[2], the chromatic number $\chi$ is at least the clique number $\omega$, and at most one more than the maximum degree $\Delta$. We seek better bounds on the chromatic number in terms of $\omega$ and $\Delta$.

Our work originates from a conjecture of Reed, who proposed that $\chi$ is essentially closer to $\omega$ than to $\Delta+1$. Our main goal is to prove this Main Conjecture for claw-free graphs, but there will be many detours along the way.

In the first chapter we cover the necessary graph colouring preliminaries. We look all the way back to Brooks' Theorem, the first step in bounding the chromatic number away from its trivial upper bound. We also discuss the polyhedral approach to graph colouring and its relationship to the theory of perfect graphs.

In Chapter 2 we trace the origin of the Main Conjecture, beginning with early attempts to strengthen Brooks' Theorem. We also propose a new, stronger variant of the Main Conjecture, which we call the Local Strengthening. We present three early results that support the Main Conjecture, and prove that their local analogues hold, thereby giving evidence in support of the Local Strengthening. The first of these states that the Local Strengthening holds for fixed $\Delta-\omega$ and sufficiently large $\Delta$. The second states that the bound proposed by the Local Strengthening holds for the fractional chromatic number; this is an unpublished result of McDiarmid. The third states that the Local Strengthening holds for graphs containing no stable set of size three. This is the first step towards bounding the chromatic number of claw-free graphs.

[2] We use standard graph theory notation, which can be found in the glossary.

# Chapter 1

# Introduction

This thesis concerns colourings of graphs[1]. Graphs arise as a natural model of networks in a variety of fields, including cellular biology, industrial optimization, communications network management, sociology, and of course computer science. In such models the edges represent the connections of the network. Edges can also represent undesirable interaction between elements of a set. For example, the vertices of a graph may represent events, and an edge between two vertices may indicate that the corresponding events cannot be run simultaneously. In such situations it is useful to partition the events into subsets of events that can be run simultaneously. This is one type of problem for which graph colouring is a very natural and practical model.

A *$k$-colouring* of a graph $G$ is an assignment of $k$ colours to the vertices of $G$ under which no two adjacent vertices receive the same colour. The *chromatic number* of $G$, denoted $\chi(G)$, is the least $k$ for which there is a $k$-colouring of $G$.

We are particularly interested in three graph invariants, $\chi(G)$ being the first. The second is the *maximum degree* of $G$, denoted by $\Delta(G)$ and equal to $\max_{v\in V(G)}\{d(v)\}$ where $d(v)$ is the degree of the vertex $v$. A *clique* in $G$ is a set of pairwise adjacent vertices. The third key invariant is the *clique number* of $G$, denoted by $\omega(G)$ and equal to the size of a largest clique in $G$.

Computing the chromatic number of a graph is a difficult (i.e. NP-complete) problem. Indeed even approximating the chromatic number is difficult [FK98]. The same is true for the clique number [Hås99], although computing the maximum degree is a simple matter.

Bounding the chromatic number is the central concern of this thesis. We investigate two approaches to doing so. The first is to obtain bounds by looking at the local structure of the graph. It is easy to see that the clique number of a graph $G$ gives a lower bound for the chromatic number: there is some set of $\omega(G)$ mutually adjacent vertices, and in any proper colouring of $G$ these vertices must each receive a different colour. Thus at least $\omega(G)$ colours are needed and $\omega(G) \leq \chi(G)$. This is an example of a local bound, i.e. a bound that is determined by the vertices of distance at most $k$ from some fixed vertex, for some fixed $k$ (in this case $k = 1$). A local upper bound for $\chi(G)$ is $\Delta(G)+1$. To see this we take $\Delta(G)+1$ colours and attempt to colour the vertices one by one in some arbitrary order. Since no vertex has more than $\Delta(G)$ neighbours, when we come to colour a vertex there will always be a colour not already appearing in its neighbourhood. So $\chi(G) \leq \Delta(G) + 1$.

Our second approach to bounding the chromatic number is to consider the fractional relaxation of the natural integer program that describes the chromatic number (the uninitiated reader will

[1] Throughout the thesis we use standard graph theory notation, which can be found in the glossary.

find definitions and some basic results in Section 1.2). The solution to this fractional relaxation is the *fractional chromatic number*, denoted by $\chi_f(G)$. Fractional colourings, interesting in their own right, sometimes give us insight to the chromatic number in the form of approximation results – we will prove such a result in Chapter 7. For some classes of graphs (e.g. line graphs and perfect graphs) it is very fruitful to consider optimization problems over a variety of polyhedra. We will consider this flavour of approach, along with the study of perfect graphs, in Chapter 3.

In Chapter 4 we consider two classes of graphs for which the chromatic number is close to the fractional chromatic number: line graphs and circular interval graphs. In later chapters we will focus on two graph classes which generalize line graphs, using structural properties of graphs in these classes to bound their chromatic number. The first is the class of *quasi-line* graphs, in which the neighbours of any vertex can be covered by two cliques. The second, even more general than quasi-line graphs, is the class of *claw-free* graphs, in which no neighbourhood contains a stable set of size three.

In Chapters 5, 6 and 7 we consider quasi-line graphs. Their structure was described by Chudnovsky and Seymour, who generalized earlier work of Chvátal, Maffray, Reed, and Sbihi on the structure of Berge quasi-line graphs. In studying quasi-line graphs we will introduce several tools and methods that will help us describe and colour claw-free graphs. In Chapter 7 we prove two bounds on the chromatic number of quasi-line graphs, thereby generalizing known results on line graphs.

In Chapter 8 we consider claw-free graphs with $\alpha \geq 4$ that are not quasi-line. Such a graph contains an odd antihole in some neighbourhood. In fact as Fouquet showed, such a graph contains an induced $C_5$ in some neighbourhood. By examining the structure around this $W_5$ we can decompose such graphs and reduce them to quasi-line graphs. In Chapter 9 we use Chudnovsky and Seymour's recent structure theorems for claw-free graphs to give a specific description of all claw-free graphs. Theirs is a unified approach, covering the case $\alpha \leq 3$ and reducing from claw-free graphs and quasi-line graphs to line graphs in one step. They also use a completely different technique for decomposing claw-free graphs with a three-colourable complement. In Chapter 10 we give a new bound on the chromatic number of claw-free graphs, in particular proving a conjecture of Reed on the relationship between $\omega$, $\Delta$, and $\chi$.

The remainder of this chapter gives a gentle introduction to graph colouring and presents the two aforementioned approaches to the problem. We say that a vertex $u$ *sees* a vertex $v$ precisely if $u$ and $v$ are adjacent, and that vertex sets $S$ and $T$ are *complete* or *joined* to one another if every possible edge between them exists. If no edge exists between them they are *anticomplete*. When we say that a graph $G$ *contains* a graph $H$ we mean that $H$ is an induced subgraph of $G$. A *hole* is a chordless cycle of length $\geq 4$ and an *antihole* is the complement of a hole.

## 1.1 Local bounds on $\chi$

We have already shown that by virtue of easy local bounds on the chromatic number, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Either one of these bounds can be tight. For example if $G$ is a complete graph, i.e. the entire vertex set of $G$ forms a clique, then $\omega(G) = \Delta(G) + 1$ and both bounds are tight. If $G$ is a cycle of odd length than $\omega(G) = 2$ and $\Delta(G) + 1 = 3$, but in any 2-colouring of $G$ we will have two adjacent vertices of the same colour, so $\chi(G) = 3 = \Delta(G) + 1$.

In 1941, Brooks [Bro41] proved that odd cycles and cliques are the only two situations in which a connected graph $G$ will have $\chi(G) = \Delta(G) + 1$. The proof we give is similar to the one given by Lovász in [Lov73].

**Theorem 1.1** (Brooks' Theorem). *For any graph $G$, $\chi(G) \leq \Delta(G)$ unless some component of $G$ is a clique of size $\Delta + 1$, or some component of $G$ is an odd cycle and $\Delta(G) = 2$.*

*Proof.* Let $G$ be a minimum counterexample. We know that graphs with $\Delta = 2$ are either bipartite or odd cycles, so $G$ must have maximum degree at least three. It is easy to see that $G$ has no cutvertex $v$, otherwise we could apply induction to $\Delta(G)$-colour $G' \cup \{v\}$ for each component $G'$ of $G - v$. By permuting the colour classes in the coloured graphs, we can ensure that the same colour appears on $v$ for each $G'$, so we can paste the coloured subgraphs together on $v$ to complete a $\Delta(G)$-colouring of $G$.

So $G$ is 2-connected with maximum degree at least three. We claim that there are three vertices $v_1$, $v_2$, and $v_n$ such that $v_n$ sees both $v_1$ and $v_2$, which do not see each other, and such that $G - \{v_1, v_2\}$ is connected. We make the easy proof an exercise for the conscientious reader. It appears in the proof of Brooks' Theorem in [Wes00].

Beginning with $v_n$, we will label the vertices of $G - \{v_1, v_2\}$ from $v_{n-1}$ down to $v_3$. To do this, we repeatedly label a vertex which already has a labeled neighbour. This is possible because $G - \{v_1, v_2\}$ is connected. This gives us an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $G$ such that for every $i < n$, $v_i$ has a neighbour $v_j$ for some $j > i$. Consequently we can give $v_1$ and $v_2$ the same colour, then greedily $\Delta(G)$-colour the remaining vertices of $G$ in order from $v_3$ to $v_n$. Since our ordering ensures that no vertex will ever have $\Delta(G)$ colours already in its neighbourhood when we want to colour it, we can complete the $\Delta(G)$-colouring safely. $\square$

This proof introduces two important ideas. First is the idea of repeated colours in a neighbourhood – in this case we begin by insisting on a repeated colour in the neighbourhood of $v_n$. Second is the idea of ensuring that in a sequential colouring, a vertex has few neighbours appering before it in the ordering. Both ideas ensure that when we want to colour a vertex, few colours will already appear in its neighbourhood.

In contrast to Brooks' characterization of graphs for which $\chi(G) = \Delta(G) + 1$, the class of graphs for which $\chi(G) = \omega(G)$ is much richer. These graphs include, for example, the class of *perfect graphs* – a graph is perfect precisely if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$.

Given a multigraph $H$, the *line graph* of $H$, denoted $G = L(H)$, is constructed as follows. Let $V(G) = E(H)$, and let two vertices in $G$ be adjacent precisely if their corresponding edges in $H$ share an endpoint. Note that the matchings of $H$ correspond naturally to the stable sets of $G$. Likewise an edge colouring of $H$ corresponds naturally to a vertex colouring of $G$. We say that $G$ is a *line graph* if there is some multigraph $H$ for which $G = L(H)$, and if there is a simple graph $G = L(H)$ we say that $G$ is a *line graph of a simple graph.* We will explore line graphs and their colourings in much more depth later in this thesis, but for now we will just state one of the first fundamental results in the area, Vizing's Theorem. The *multiplicity* of a multigraph is the maximum number of edges between two vertices.

**Theorem 1.2** (Vizing's Theorem [Viz64]). *If $G$ is the line graph of a multigraph $H$ of multiplicity $\mu$, then $\chi(G) \leq \Delta(G) + \mu$.*

**Corollary 1.3.** *If $G$ is the line graph of a simple graph $H$, then $\chi(G) \leq \omega(G) + 1$.*

For perfect graphs and line graphs of simple graphs, $\chi(G)$ is always near $\omega(G)$ but may be far from $\Delta(G)$. Is $\chi(G)$ essentially always as close to $\omega(G)$ as it is to $\Delta(G) + 1$? We explore this question throughout the thesis, particularly in Chapters 2, 4, 7, and 10. Brooks' Theorem tells us that if $\Delta(G) > 2$, then the bound of $\Delta(G) + 1$ is tight for $\chi(G)$ only if the bound $\omega(G)$ is tight.

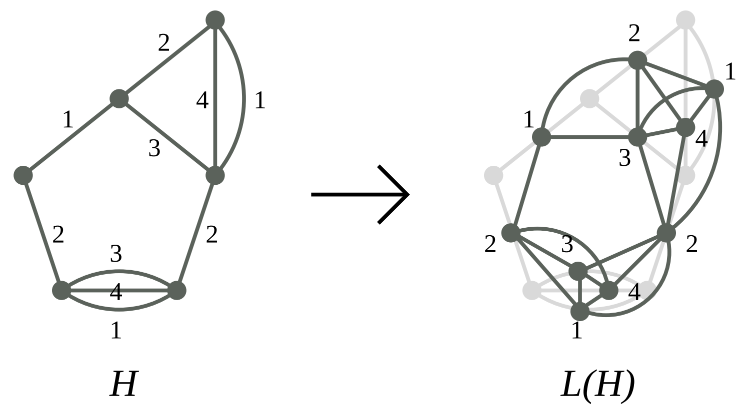


Figure 1.1: A multigraph $H$ and its line graph $G = L(H)$. An edge colouring of $H$ corresponds to a vertex colouring of $G$.

### 1.1.1 Sequential colouring

We proved earlier that any graph $G$ can be coloured using $\Delta(G) + 1$ colours regardless of the order in which we process the vertices. As we showed in the proof of Brooks' Theorem, we can hope to improve the situation by carefully choosing the order in which we colour the vertices. Ideally we would like to ensure that every uncoloured vertex has few coloured neighbours. To this end we define the *colouring number* of a graph $G$, denoted by $col(G)$, as the maximum of $\delta(H) + 1$ over all subgraphs $H$ of $G$. We can always colour the vertices of $G$ in order such that no uncoloured vertex has more than $col(G)$ coloured neighbours when we come to colour it.

**Lemma 1.4.** *For any graph $G$, $\chi(G) \leq col(G)$.*

*Proof.* Choose an ordering $v_1, \ldots, v_n$ of the vertices as follows, Having chosen $v_j$ for $n \geq j > i$, let $v_i$ be a minimum degree vertex in $G - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. Now $col(G)$-colour the vertices in order from $v_1$ onwards, noting that if $v_i$ already has $col(G)$ coloured neighbours then the graph $G - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ has minimum degree at least $col(G)$, contradicting the definition of the colouring number. □

This early result was proved independently by several people, including Erdős and Hajnal [EH66] and Szekeres and Wilf [SW68]. One special application of this idea is the following: If a graph $G$ has chromatic number $k$ and a vertex $v$ with $d(v) < k - 1$, then $G - v$ also has chromatic number $k$. This is because a $(k-1)$-colouring of $G - v$ will never result in $v$ having $k - 1$ coloured neighbours, so we can always extend such a colouring of $G - v$ to a $(k-1)$-colouring of $G$. We will use this argument repeatedly throughout the thesis. A good example appears in the proof of Theorem III.2, specifically in the proof of Lemma 7.5.

Brooks' Theorem tells us which graphs satisfy $\chi(G) = \Delta(G) + 1$ and which graphs satisfy $\chi(H) = \Delta(H) + 1$ for every induced subgraph $H$ of $G$ ($G$ must be a clique). In contrast it is difficult (i.e. NP-complete) to decide whether or not $\chi(G) = \omega(G)$, but we can determine whether a graph is perfect in polynomial time [CCL+05]. This is done through a structural characterization of perfect graphs; we discuss this result in Section 3.6.

It is natural to ask the same question about the colouring number. It is not known whether we can efficiently decide if $\chi(G) = col(G)$ for any graph, and no structural characterization for such graphs is known. However along these lines Markossian, Gasparian and Reed introduced the class of *$\beta$-perfect graphs* [MGR96]. A graph is $\beta$-perfect precisely if $\chi(H) = col(H)$ for every induced subgraph $H$ of $G$ ($\beta(G)$ is sometimes used to denote the colouring number of $G$). Can we characterize the structure of $\beta$-perfect graphs? It is easy to see that no $\beta$-perfect graph contains an even hole. An *even hat* (resp. *odd hat*) is an even (odd) cycle of length $\geq 4$ with exactly one chord, which makes a triangle with two edges of the cycle. Markossian, Gasparian and Reed provide two structure theorems [MGR96], although it is still not known whether $\beta$-perfect graphs can be recognized in polynomial time.

**Theorem 1.5.** *If a graph $G$ contains no even hole or even hat, then $G$ is $\beta$-perfect.*

**Theorem 1.6.** *A graph $G$ and its complement are both $\beta$-perfect precisely if neither $G$ nor $\overline{G}$ contains an even hole.*

Theorem 1.5 was strengthened by de Figueiredo and Vušković so that only even hats on four or six vertices need to be restricted [dFV00]; Keijsper and Tewes offer more complicated strengthenings [KT02]. Theorem 1.6 is an interesting analogue of the Strong Perfect Graph Theorem, which we present in the next section.

## 1.2 Fractional colouring and perfect graphs

The chromatic number is the least number of colours needed to properly colour the vertices of a graph. Since each colour class in a proper colouring is a stable set, the chromatic number is also the least number of stable sets needed to cover the vertex set. Therefore we consider the following integer program, where $\mathcal{S}(G)$ denotes the family of stable sets in a graph, and for $S \in \mathcal{S}(G)$, $w_S$ is 1 precisely if $S$ is a colour class. (For definitions and preliminaries on linear programming, see [Chv83].)

$$\begin{aligned} \text{minimize} \quad & \sum_{S \in \mathcal{S}(G)} w_S \\ \text{subject to} \quad & \forall v \in V(G), \sum_{S \ni v} w_S = 1 \\ \text{and} \quad & \forall S \in \mathcal{S}(G), w_S \in \{0,1\} \end{aligned} \tag{1.1}$$

In this case we insist that we cover each vertex with precisely one stable set. But it is often useful to consider covering a vertex with a combination of stable sets. Thus we consider the fractional relaxation (1.2) of the above integer linear program (1.1).

$$\begin{aligned} \text{minimize} \quad & \sum_{S \in \mathcal{S}(G)} w_S \\ \text{subject to} \quad & \forall v \in V(G), \sum_{S \ni v} w_S \geq 1 \\ \text{and} \quad & \forall S \in \mathcal{S}(G), w_S \in [0,1] \end{aligned} \tag{1.2}$$

A weighting of stable sets that satisfies the conditions of the fractional relaxation is a *fractional colouring* of $G$, and if $\sum_{S \in \mathcal{S}(G)} w_S = c$ we call it a *fractional $c$-colouring*. The *fractional chromatic number* of $G$, denoted by $\chi_f(G)$, is the least $c$ for which there is a fractional $c$-colouring of $G$. Since we are dealing with the fractional relaxation of a minimization problem, we always have

$\chi_f(G) \leq \chi(G)$. In Chapter 3 we will prove that for any graph, $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$. These are most definitely not local bounds in any sense. Since $\chi_f(G) \leq \chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$ and it is NP-hard to approximate $\chi(G)$ to within $n^{1-\epsilon}$ for any $\epsilon > 0$, we can see that computing the fractional chromatic number of a graph is also hard, since doing so would give us a $(\log n)$-approximation to $\chi(G)$.

One way to approach (1.2) is to study its dual. In this case the dual linear program maximizes the total weight on vertices so that no stable set has total weight more than 1:

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & \forall S \in \mathcal{S}(G), \sum_{v \in V} x_v \leq 1 \\ \text{and} \quad & \forall v \in V, x_v \in [0,1] \end{aligned} \tag{1.3}$$

Let $\omega_f(G)$ denote the total weight of an optimum dual solution. LP duality tells us that $\omega_f(G) = \chi_f(G)$. So putting weight 1 on every vertex of a maximum clique and 0 elsewhere gives the bound $\omega(G) \leq \omega_f(G) = \chi_f(G)$. This is a trivial bound and we do not need duality to see it. However, we will later exploit this duality to make nontrivial statements about constructing fractional colourings of claw-free graphs.

There are several important polyhedra related to this linear program – we will discuss them in Chapter 3. Its feasible region is the *fractional clique polytope*, which is the *fractional stable set polytope* of $\overline{G}$. The fractional stable set polytope of $G$ contains the *stable set polytope*, which is the convex hull of the incidence vectors of the stable sets of $G$.

One class of graphs for which the stable set polytope is well understood is line graphs, in which stable sets correspond to matchings in the base multigraph – the characterization of the stable set polytope of line graphs is a fundamental result of Edmonds [Edm65a]. We have a good understanding of the stable set polytopes of perfect graphs as well: In Chapter 3 we will show that $G$ is perfect precisely if the fractional stable set polytope is equal to the stable set polytope.

Perfect graphs have been an important area of research from the 1960s onwards, thanks in large part to their connection to communication theory and two conjectures posed by Berge [Ber61] (translated to English in [BR01]):

**Conjecture 1.7** (The Weak Perfect Graph Conjecture)**.** *A graph $G$ is perfect precisely if its complement $\overline{G}$ is perfect.*

A graph is called *Berge* if it contains neither an odd hole nor an odd antihole.

**Conjecture 1.8** (The Strong Perfect Graph Conjecture)**.** *A graph $G$ is perfect precisely if it is Berge.*

The key to the proof of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour, and Thomas [CRST06] is the fact that any Berge graph either lies in some well-understood graph class or is pieced together from smaller graphs in some well-understood way. This is the same angle of attack we use for proving the results on quasi-line and claw-free graphs.

In Chapter 3 we will discuss how both of Berge's conjectures were proved (thirty years apart) and how the theory of perfect graphs relates to fractional colourings, polyhedra and optimization, structural decompositions, and claw-free graphs.

The Strong Perfect Graph Theorem offers a structural characterization of a class of graphs for which $\chi_f(G) = \chi(G)$. By Vizing's Theorem, the fractional and integer chromatic numbers must be within 1 of each other for line graphs of simple graphs. The same is not known to be true for line graphs of multigraphs, but would be implied by the Goldberg-Seymour Conjecture [Gol73]. Kahn [Kah96] proved that they agree asymptotically using properties of the stable set polytope of line graphs.

# Chapter 2

# $\omega$, $\Delta$, and $\chi$: Conjectures Old and New

We consider the following conjecture of Reed,

**Conjecture 2.1** ([Ree98])**.** *For any graph $G$ and integer $k \geq 1$, if $G$ contains no clique of size greater than $\Delta(G) + 1 - 2k$, then $\chi(G) \leq \Delta(G) + 1 - k$.*

and a strengthening which is proposed for the first time here,

**Conjecture 2.2.** *For any graph $G$ and integer $c \geq 1$, if every vertex of $v$ satisfies $d(v)+1+\omega(v) \leq 2c$, then $G$ is $c$-colourable.*

where $\omega(v)$ denotes the size of the largest clique containing $v$. We can see that the second conjecture is stronger by replacing $d(v) + 1 + \omega(v)$ with $\Delta(G) + 1 + \omega(G)$.

The two strongest pieces of evidence for Reed's conjecture are that a fractional variant is true, and for any $k$ there is a $\Delta_k$ for which the conjecture holds whenever $\Delta(G) \geq \Delta_k$. Further supporting the conjecture is the fact that when $\omega(G)$ is bounded, the ratio $\chi(G)/\Delta(G)$ approaches zero as $\Delta(G)$ increases.

In this chapter we discuss these results and extend them by proving analogues for the local variant of Reed's conjecture. We then present related results and a further strengthening of the two conjectures, which we prove is false. We begin by presenting the motivation for Conjecture 2.1.

## 2.1 Beyond Brooks' Theorem

From Brooks' Theorem, we know that we can always colour $G$ using at most $\Delta(G)$ colours as long as no connected component of $G$ is a clique of size $\Delta(G) + 1$ (or a cycle of odd length, if $\Delta(G) = 2$). This gives a complete characterization of graphs for which $\chi(G) = \Delta(G) + 1$. In an attempt to characterize graphs for which $\chi(G) \geq \Delta(G)$, Borodin and Kostochka proved that if $\omega(G) \leq \Delta(G)/2$ then $\chi(G) \leq \Delta(G) - 1$, then made the following conjecture:

**Borodin-Kostochka Conjecture** ([BK77])**.** *If a graph $G$ with $\Delta(G) \geq 9$ contains no clique of size $\Delta(G)$, then $\chi(G) \leq \Delta(G) - 1$.*

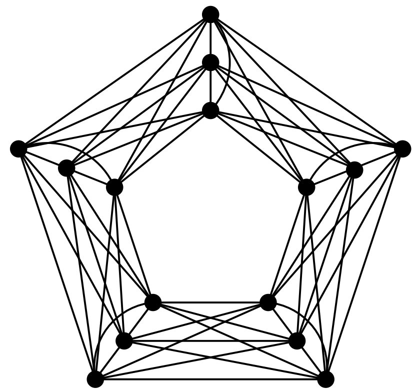

Figure 2.1: A graph $G$ with $\omega(G) = 6$, $\Delta(G) = 8$, and $\chi(G) = \Delta(G)$.

Figure 2.1 illustrates that the Borodin-Kostochka Conjecture is the strongest variant of this conjecture which is not false. Beutelspacher and Hering independently made the weaker conjecture that there is some $\Delta_2$ for which $\chi(G) \leq \Delta(G) - 1$ provided $\Delta(G) \geq \Delta_2$ and $\omega(G) \leq \Delta(G) - 1$ [BH84]. Reed proved the Beutelspacher-Hering Conjecture for $\Delta_2 = 10^{14}$ [Ree99].

This result suggests the possibility that for all $k$ there may be a $\Delta_k$ such that $G$ has $\chi(G) \leq \Delta(G)+1-k$ provided that $\Delta(G) \geq \Delta_k$ and $\omega(G) \leq \Delta(G)+1-k$. However, this is easily disproved for $k = 3$. To do so, we can join a $(\Delta - 4)$-clique to a $C_5$ to get a graph with maximum degree $\Delta$, clique number $\Delta - 2$, and chromatic number $\Delta - 1$. So as we move the upper bound on $\omega(G)$ away from $\Delta(G)+1$, we cannot hope that $\chi(G)$ will move away from $\Delta(G)+1$ at the same rate. But we can hope that it moves away from $\Delta(G) + 1$ at some rate for $k > 2$. An example in [Ree98] shows that a rate of $\frac{1}{2}$ would be best possible, and Reed gave evidence that it may in fact be the case:

**Theorem 2.3** ([Ree98]). *For any positive integer $k$ there is some constant $\Delta_k$ such that any graph $G$ with $\Delta(G) \geq \Delta_k$ and $\omega(G) \leq \Delta(G) + 1 - 2k$ will have chromatic number at most $\Delta(G) + 1 - k$.*

This result is actually a corollary of a more significant theorem from the same paper, which states that the Conjecture 2.1 holds provided that $\omega(G)$ and $\Delta(G)+1$ are sufficiently close together:

**Theorem 2.4** ([Ree98]). *There is a universal positive constant $\eta$ such that if $\omega(G) \geq (1-\eta)(\Delta(G)+1)$ then $\chi(G) \leq \frac{1}{2}(\Delta(G) + 1 + \omega(G))$.*

This result also tells us that we can find a nontrivial upper bound for $\chi(G)$ by taking a convex combination of $\omega(G)$ and $\Delta(G) + 1$:

**Corollary 2.5** (Reed). *There is a universal positive constant $a$ such that $\chi(G) \leq (1 - a)(\Delta(G) + 1) + a\omega(G)$.*

Following this thread, Reed posed Conjecture 2.1, which is equivalent to saying that $a$ can be as high as $\frac{1}{2}$ if we round up. We therefore define $\gamma(G)$ as $\lceil \frac{1}{2}(\Delta(G)+1+\omega(G)) \rceil$ and restate Conjecture 2.1, which we call the Main Conjecture:

**Main Conjecture.** *For any graph $G$, $\chi(G) \leq \gamma(G)$.*

Accordingly, we define $\gamma_l(v)$ as $\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$ and we define $\gamma_l(G)$ as $\max_{v\in G} \gamma_l(v)$. Equivalently, $\gamma_l(v) = \gamma(G[\tilde{N}(v)])$ and $\gamma_l(G) = \max_{v\in G} \gamma(G[\tilde{N}(v)])$. Thus we can restate the local variant of the Main Conjecture, which we call the Local Strengthening:

**Local Strengthening.** *For any graph $G$, $\chi(G) \leq \gamma_l(G)$.*

Theorem 2.3 is the first major piece of evidence for the Main Conjecture. We now prove that we can strengthen it to find evidence supporting the Local Strengthening.

**Theorem 2.6.** *For any positive integer $k$ there is some constant $\Delta'_k$ such that any graph $G$ with $\Delta(G) \geq \Delta'_k$ in which every vertex $v$ satisfies $d(v)+1+\omega(v) \leq 2(\Delta(G)+1-k)$ will have chromatic number at most $\Delta(G)+1-k$.*

To prove this we use a result of Molloy and Reed that guarantees the existence of a small $\chi(G)$-chromatic subgraph of $G$ under certain conditions.

**Theorem 2.7** (Molloy and Reed [MR01])**.** *There is an absolute constant $\Delta_0$ such that for any $\Delta \geq \Delta_0$ and $k \leq \sqrt{\Delta} - 3$, if $G$ has maximum degree $\Delta$ and $\chi(G) > \Delta - k$ then $G$ contains a subgraph $H$ such that: (i) $|H| \leq \Delta + 1$; (ii) $\chi(H) = \Delta + 1 - k$.*

**Corollary 2.8.** *There is an absolute constant $\Delta_0 \geq 4$ such that for any $\Delta \geq \Delta_0$ and $k \leq \frac{1}{2}\sqrt{\Delta}$, if $G$ has maximum degree $\Delta$ and $\chi(G) > \Delta - k$ then $G$ contains a subgraph $H$ with a universal vertex such that $\chi(H) = \Delta - k + 1$.*

*Proof.* We claim that given our further constraint on $k$, there is a subgraph $H$ guaranteed by Theorem 2.7 that contains a universal vertex. To this end, assume that $H$ is minimal and take an optimal colouring of $H$. The singleton colour classes make up a clique $C$ in $H$, and since $|H| \leq \Delta+1$ we can see that $|C| \geq \Delta+1-2k$ and there are at most $2k$ vertices outside $C$. Since $\chi(H) = \Delta-k+1$ and by minimality $H$ is vertex-critical, every vertex in $H - C$ has at most $k$ non-neighbours in $H$. Thus the number of universal vertices in $C$ is at least $\Delta + 1 - 2k - 2k^2 > \Delta + 1 - \sqrt{\Delta} - \frac{1}{2}\Delta > 0$, so there is at least one universal vertex in $C$. □

Now we can easily prove Theorem 2.6. The key to the proof is the fact that if $v$ is a universal vertex in a graph $G$, then $\Delta(G) + 1 + \omega(G) = d(v) + 1 + \omega(v)$.

*Proof of 2.6.* Let $\Delta_k$ and $\Delta_0$ be as in Theorems 2.3 and 2.7 respectively, and let $\Delta'_k = 2 \cdot \max\{\Delta_k, \Delta_0, 2k^2\}$. Suppose $G$ is a counterexample to the theorem, i.e. $\chi(G) > \Delta(G) + 1 - k$, but $\Delta(G) \geq \Delta'_k$ and every vertex $v$ of $G$ satisfies $d(v) + 1 + \omega(v) \leq 2(\Delta(G) + 1 - k)$.

By Corollary 2.8 there is a $\chi(G)$-chromatic subgraph $H$ of $G$ with a universal vertex $v$. Let $k' = k + \Delta(H) - \Delta(G)$; then $k' \leq k$ and $\chi(H) > \Delta(H) + 1 - k'$. By our choice of $\Delta'_k$, $\chi(G) > \Delta_k \geq \Delta_{k'}$, therefore $H$ satisfies the conditions of Theorem 2.3. Thus $\omega(H) > \Delta(H) + 1 - 2k'$ because $\chi(H) > \Delta(H) + 1 - k'$. Since $v$ is universal it is in every maximum clique of $H$, so $d_H(v)+1+\omega_H(v) > 2(\Delta(H)+1-k')$. Therefore $d_G(v)+1+\omega_G(v) > 2(\Delta(G)+1-k)$, contradicting our choice of $G$. □

Next we will give our second piece of evidence: the fractional analogues of the Main Conjecture and Local Strengthening are true.

## 2.2 Fractionally, $\chi \leq \gamma$

One of the most encouraging pieces of evidence for the Main Conjecture is the fact that it holds for the fractional chromatic number. We use $\gamma'(G)$ to denote $\frac{1}{2}(\Delta(G) + 1 + \omega(G))$, i.e. the unrounded version of $\gamma(G)$. Similarly we use $\gamma'_l(G)$ to denote $max_{v \in V}\{\frac{1}{2}(d(v) + 1 + \omega(v)\}$, the unrounded version of $\gamma_l(G)$.

**Theorem 2.9** ([MR00]). *For any graph $G$, $\chi_f(G) \leq \gamma'(G)$.*

One can easily see that $\chi_f(G) \geq \frac{n}{\alpha(G)}$, thus the fractional version of the Main Conjecture implies that every graph $G$ contains a stable set of size at least $\frac{2n}{\Delta(G)+1+\omega(G)}$, i.e. $\frac{n}{\gamma(G)}$. This weaker result was proven much earlier by Fajtlowicz [Faj78, Faj84]. Here we prove that the Local Strengthening also holds fractionally. This is an unpublished result of McDiarmid whose proof closely follows the proof of Theorem 2.9 in Chapter 21 of [MR00]; it is left as an exercise in the book.

**Theorem 2.10.** *For any graph $G$, $\chi_f(G) \leq \gamma_l'(G)$.*

The key to this result is the following lemma, which tells us that a random maximum stable set either hits a vertex or two of its neighbours with sufficiently high probability. It strengthens 21.10 from [MR00], replacing $\omega(G)$ with $\omega(v)$.

**Lemma 2.11.** *Let $G$ be a graph and let $S$ be a maximum stable set of $G$ chosen uniformly at random. Then for every vertex $v$ of $G$,*

$$\mathbf{E}(|S \cap N(v)|) \geq 2 - (\omega(v)+1)\mathbf{Pr}(v \in S). \tag{2.1}$$

We will prove this lemma later in the section. As a warm-up to the proof of Theorem 2.10 we will use Lemma 2.11 to easily prove the natural strengthening of Fajtlowicz's result:

**Lemma 2.12.** *For any graph $G$, $\alpha(G) \geq \frac{n}{\gamma_l'(G)}$.*

*Proof.* Let $S$ be a uniformly random maximum stable set in $G$. Rearranging (2.1) and summing over all vertices, we get

$$\begin{aligned}
2n &\leq \sum_v (\omega_v+1)\mathbf{Pr}(v \in S) + \sum_v \mathbf{E}(|S \cap N(v)|) &(2.2)\\
&= \sum_{v \in V} (\omega_v+1)\mathbf{Pr}(v \in S) + \sum_v \sum_{u \sim v} \mathbf{Pr}(u \in S) &(2.3)\\
&= \sum_{v \in V} (\omega_v+1)\mathbf{Pr}(v \in S) + \sum_{u \in V} d(u)\mathbf{Pr}(u \in S) &(2.4)\\
&= \sum_{v \in V} (\omega_v+1+d(v))\mathbf{Pr}(v \in S) &(2.5)\\
&\leq 2\gamma_l' \sum_{v \in V} \mathbf{Pr}(v \in S) &(2.6)
\end{aligned}$$

Since the expected size of a maximum stable set is precisely the sum of the probabilities of each vertex being in the set, we get $\mathbf{E}(|S|) \geq n/\gamma_l'$. The theorem follows. □

If every vertex was in the same number of maximum stable sets, then this lemma would immediately imply the desired result, Theorem 2.10. We could find a suitable fractional colouring by placing equal weight on every maximum stable set and no weight on smaller stable sets. However, in general we do not have this desirable condition. For example in the star graph $K_{1,k}$ for $k > 1$, one vertex is in no maximum stable sets, and every other vertex is in every maximum stable set (although there is only one). So instead we must use Lemma 2.11 in an iterative fashion. The proof of Theorem 2.9 given in [MR00] uses the same method but uses weaker supporting lemmas.

*Proof of Theorem 2.10.* We fractionally colour $G$ using the following iterative method.

1. Set $w_S = 0$ for every $S \in \mathcal{S}$. Set $G_0 = G$. Set $i = 0$.

   Set $T = 0$. $T$ stands for total weight used.

   For each $v \in V$, set $wo_v = 0$ ($wo$ stands for weight on).

2. If $V(G_i) = \emptyset$ or $T = \gamma'_l$ then stop.

3. For each vertex $v$ of $G_i$, let $p_i(v)$ be the probability that $v$ is in a uniformly random maximum stable set of $G_i$. Set $low = \min\{\frac{1-wo_v}{p_i(v)} | v \in V(G_i)\}$. Set $val_i = \min(low, \gamma'_l - T)$.

4. Let $\mathcal{S}_i$ be the set of maximum stable sets of $G_i$. For each stable set in $\mathcal{S}_i$, increase $w_S$ by $\frac{val_i}{|\mathcal{S}_i|}$. For each vertex $v$ of $G_i$, increase $wo_v$ by $p_i(v)val_i$. Increase $T$ by $val_i$.

5. Let $G_{i+1}$ be the graph induced by those vertices $v$ which satisfy $wo_v < 1$. Increment $i$ and go to Step 2.

Our choice of $val_i$ ensures two things: that $T$ never exceeds $\gamma'_l$, and that if the $i$th iteration is not the last, then $V(G_{i+1})$ is properly contained in $V(G_i)$. Thus the algorithm must terminate.

We claim that at the end of the procedure, the $w_S$ yield a fractional $\gamma'_l$-colouring. It is easy to show by induction that at the end of each iteration and for every $v \in V$, $wo_v = \sum_{\{S \in \mathcal{S} | v \in S\}} w_S$ and $T = \sum_{S \in \mathcal{S}} w_S$. The definitions of $low$ and $val_i$ ensure that no $wo_v$ is ever more than 1. We stop if $V(G_i) = \emptyset$ or $T = \gamma'_l$; in the first case we know that we have the desired fractional colouring. We must now show that the same is true in the second case. It suffices to show that in this case, each $wo_v = 1$.

So assume that for some $v$ we have $wo_v < 1$ when we complete the process. For each vertex $u$ and iteration $i$, denote by $a_i(u)$ the amount by which $wo_u$ was augmented in iteration $i$, i.e. $a_i(u) = val_i p_i(u)$. By Lemma 2.11, $\mathbf{E}(|S \cap N(v)|) \geq 2 - (\omega(v) + 1)\mathbf{Pr}(v \in S)$. It follows that since $v$ was in each $G_i$, we have, for each $i$,

$$\sum_{u \in N(v)} a_i(u) \geq 2(val_i) - (\omega(v) + 1)a_i(v). \tag{2.7}$$

Summing over the iterations, we get

$$\sum_{u \in N(v)} wo_u \geq 2T - (\omega(v) + 1)wo_v > (\omega(v) + d(v) + 1) - (\omega(v) + 1) = d(v). \tag{2.8}$$

But since $wo_u \leq 1$ for every $u$, we cannot have $\sum_{u \in N(v)} wo_u > d(v)$. Thus we obtain a contradiction and so $wo_v = 1$. This completes the proof. □

It remains for us to prove Lemma 2.11. We will first prove one intermediate result.

**Lemma 2.13.** *If $S$ is chosen uniformly at random from the maximum stable sets of $G$, then for every vertex $v$ of $G$ we have*

$$\mathbf{Pr}(|S \cap N(v)| = 1) \leq (\omega(v) - 1)\mathbf{Pr}(v \in S). \tag{2.9}$$

*Proof.* We fix $v$ and condition on our choice of $S' = S - \bar{N}(v)$, showing that the inequality holds no matter what the value of $S'$ is. So fix some $R \in \mathcal{S}(V - \bar{N}(v))$ that can be extended to a maximum stable set of $G$ by adding vertices in $\bar{N}(v)$. To prove the lemma, it is enough to show that for any choice of $R$,

$$\mathbf{Pr}(|S \cap N(v)| = 1 | S' = R) \leq (\omega(v) - 1)\mathbf{Pr}(v \in S | S' = R). \tag{2.10}$$

So fix a choice of $R$ and let $W = \bar{N}(v) - N(R)$. First suppose that $W$ is not a clique. Then we know that $|R| \leq \alpha(G) - 2$, and since $v$ is universal in $R$ we have $|S \cap \bar{N}(v)| \geq 2$ if $S' = R$, since $S$ is a maximum stable set. Now suppose that $W$ is a clique; we know that $|W| \leq \omega(v)$ since $v \in W$. Since $R$ can be extended to a maximum stable set, $|R| = \alpha(G) - 1$ and $S$ is equally likely to be $R + u$ for any vertex $u \in \bar{N}(v)$. If $u = v$ then $|S \cap \bar{N}(v)| = 0$, so it follows that

$$\mathbf{Pr}(v \in S | S' = R) = \frac{1}{|W|} \tag{2.11}$$

and

$$\mathbf{Pr}(|S \cap N(v)| = 1 | S' = R) = \frac{|W| - 1}{|W|} \leq \frac{\omega(v) - 1}{|W|}. \tag{2.12}$$

The lemma follows. □

Proving Lemma 2.11 from here is a simple matter.

*Proof of Lemma 2.11.* Since $S$ is a maximum (and therefore maxim*al*) stable set, $\mathbf{Pr}(v \in S) = \mathbf{Pr}(S \cap N(v) = \emptyset)$. So since $|S \cap N(v)|$ is a nonnegative integer variable, we have $\mathbf{E}(|S \cap N(v)|) \geq 2 - \mathbf{Pr}(|S \cap N(v)| = 1) - 2\mathbf{Pr}(|S \cap N(v)| = 0)$. The result now follows from Lemma 2.13. □

## 2.3 Graphs with small clique number

We have seen that the Main Conjecture holds when $\omega(G)$ is close to $\Delta(G) + 1$. We now consider the other end of the spectrum, when $\omega(G)$ is very small. When $\omega \in \{0, 1\}$, $\chi = \omega$ and the result is trivial. However, the conjecture is not known to hold when $\omega = 2$.

This special case is equivalent to the statement that any triangle-free graph has chromatic number at most $\frac{\Delta}{2} + 2$. A classical result of Lovász [Lov66] implies that we can partition the vertices of a graph $G$ into $m$ parts, each inducing a subgraph of maximum degree at most three, if $4m \geq \Delta(G) + 1$. If $G$ is triangle-free, then by Brooks' Theorem each induced subgraph is 3-colourable; this implies that $\chi(G) \leq 3m \leq 3\lceil(\Delta(G) + 1)/4\rceil$. Kostochka (see [JT95], Section 4.6) proved that any triangle-free graph $G$ satisfies $\chi(G) \leq \frac{2}{3}\Delta(G) + 2$. He also proved that for every $\Delta > 4$ there is a $g_\Delta$ such that any graph with maximum degree $\Delta$ and girth at least $g_\Delta$ is $(\frac{\Delta}{2} + 2)$-colourable.

In a stronger asymptotic result, Johansson [Joh96] (also see [MR00]) proved that triangle-free graphs have chromatic number at most $O(\frac{\Delta(G)}{\log \Delta(G)})$, extending a result of Kim [Kim95] for graphs with girth at least five. Asymptotically, this is a much stronger bound than is required by the Main Conjecture. However, the Main Conjecture is still open for triangle-free graphs of small maximum degree greater than four. In unpublished work, Johansson [Joh] claimed a proof that for any fixed $k$, graphs containing no $k$-clique satisfy $\chi(G) \leq O(\frac{\Delta(G) \log\log \Delta(G)}{\log \Delta(G)})$. Johansson's results give equal support to the Main Conjecture and Local Strengthening, since as $\Delta(G)$ grows, both $\gamma(G)$ and $\gamma_l(G)$ approach $\frac{1}{2}\Delta(G)$ asymptotically when the clique number is fixed.

In contrast to these upper bounds on $\chi$, we have known for a long time that $\chi$ can be arbitrarily high even when $\omega$ is bounded. In 1955, Mycielski showed how to construct a sequence of graphs $G_i$ for $i \geq 1$ such that $\chi(G_i) = i$ while $\omega(G_i) \leq 2$ [Myc55]. A few years later, Erdős gave a probabilistic proof of the existence of $k$-chromatic graphs with girth at least $g$ for any fixed $g$ and $k$ [Erd59].

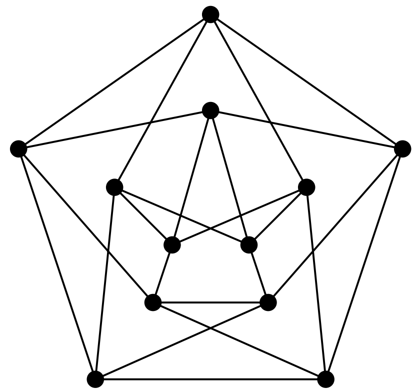

Figure 2.2: The *Chvátal graph*, a triangle-free 4-regular 4-chromatic graph.

## 2.4 Tightness of both conjectures

We now give some examples to show the Main Conjecture and the Local Strengthening are tight. That is, the round-up in $\gamma$ and $\gamma_l$ are necessary. Equivalently, it is necessary to insist that $k$ in Conjecture 2.1 and $c$ in Conjecture 2.2 are integers. Odd cycles are one class of graphs for which the round-up is necessary. It remains necessary even if $\Delta(G) > 2$: The 4-regular, 4-chromatic, triangle-free *Chvátal graph* [Chv70], shown in Figure 2.2, is one example showing this. Kostochka [Kos84] gives an example for arbitrarily high $\Delta$: take the line graph of a 5-cycle in which there are $k$ copies of every edge for some odd $k$. This graph has $\Delta(G) + 1 = 3k$, $\omega(G) = 2k$, and $\chi(G) = \lceil 5k/2 \rceil = \lceil n/\alpha \rceil$; Figure 2.1 actually shows this graph for $k = 3$. It follows that there is no $a > \frac{1}{2}$ such that $\chi \leq \lceil (1-a)(\Delta+1) + a\omega \rceil$ for all graphs. The aforementioned example in [Ree98], which has $\alpha = 2$, gives the same upper bound on $a$.

## 2.5 Early work on the Main Conjecture

Attacks on the Main Conjecture tend to fall into two categories: attempts to bound the chromatic number away from $\Delta(G) + 1$ and towards $\omega(G)$ for all graphs (e.g. Theorem 2.4), and attempts to prove the conjecture outright for restricted classes of graphs. In this thesis we present several results of the latter type. In Chapters 4, 7, and 10 we prove that the Main Conjecture holds for three increasingly general classes of graphs: line graphs (in joint work with Reed and Vetta [KRV07], quasi-line graphs (in joint work with Reed [KR08b]), and claw-free graphs (in joint work with Reed).

Randerath and Schiermeyer proved the conjecture for several classes of graphs, including graphs on at most 12 vertices and graphs for which $\Delta(G) \geq n - \alpha(G)$ [RS06, Sch06, Sch07b]. Rabern improved this latter result [Rab08]:

**Theorem 2.14.** *Let $G$ be a graph satisfying $\Delta(G) \geq n + 2 - \alpha(G) - \sqrt{n + 5 - \alpha(G)}$. Then $\chi(G) \leq \gamma(G)$.*

He also proved that $\chi(G) \leq \gamma(G)$ for any graph with a disconnected complement and any graph with $\chi(G) > \lceil \frac{n}{2} \rceil$ or $\alpha(G) = 2$. We now prove that the Local Strengthening also holds whenever

$\alpha(G) = 2$. Our proof is similar to but independent of Rabern's proof that the Main Conjecture holds when $\alpha(G) = 2$.

**Theorem 2.15.** *Let $G$ be a graph with $\alpha(G) = 2$. Then $\chi(G) \leq \gamma_l(G)$.*

Before proving this theorem we need some background on matchings in graphs. If $\alpha(G) = 2$, then an optimal colouring of $G$ corresponds to a maximum packing of vertex-disjoint stable sets of size two, which in turn corresponds to a maximum matching in $\overline{G}$. We say that a matching *hits* a vertex if the vertex is an endpoint of an edge in the matching, otherwise the matching *misses* the vertex. A *near-perfect matching* is a matching that misses exactly one vertex. The Edmonds-Gallai structure theorem [Edm65b, Gal59] describes the structure of a graph in terms of its maximum matchings (also see [LP86, Pul87]). We say that a connected component of a graph is *odd* if it contains an odd number of vertices, otherwise it is *even*.

**Theorem 2.16** (Edmonds-Gallai structure theorem)**.** *For any graph $G$, let $Y$ be the set of vertices not hit by every maximum matching in $G$, and let $X = N(Y) \setminus Y$. Then every even component of $G-X$ has a perfect matching, and every odd component has, for each of its vertices $v$, a near-perfect matching that misses $v$. Furthermore the odd components of $G - X$ have union $Y$.*

We now use this theorem to prove Theorem 2.15.

*Proof of Theorem 2.15.* Let $G$ be a minimum counterexample to the theorem. By minimality of $G$, removing any vertex will reduce the chromatic number. Therefore every vertex $v$ is missed by some maximum matching of $\overline{G}$, otherwise we would have $\chi(G - v) = \chi(G)$.

By the Edmonds-Gallai structure theorem, either $\overline{G}$ is not connected or it has a near-perfect matching. In the first case, $V(G)$ can be partitioned into nonempty $V_1$ and $V_2$ such that every possible edge between $V_1$ and $V_2$ exists. It is easy to confirm that $\chi(G) = \chi(G[V_1]) + \chi(G[V_2]) \leq \gamma_l(G[V_1]) + \gamma_l(G[V_2]) \leq \gamma_l(G)$, the middle inequality following from the minimality of $G$.

In the second case, $n$ is odd and $\chi(G) = \lceil \frac{n}{2} \rceil$. Since $\chi_f(G) \geq \frac{n}{\alpha(G)}$, we have $\chi(G) = \lceil \chi_f(G) \rceil$. By Theorem 2.10, $\chi(G) = \lceil \chi_f(G) \rceil \leq \lceil \gamma'_l(G) \rceil = \gamma_l(G)$. This proves the theorem. □

It is not hard to prove the case $\chi(G) = \lceil \frac{n}{2} \rceil$ without using Theorem 2.10. But applying the fractional result in this way is a useful technique which we will use again in proving the Main Conjecture for line graphs and the Local Strengthening for circular interval graphs in Chapter 4.

We will now provide some insight as to why the Local Strengthening is useful when considering the Main Conjecture, and why it is in a certain sense the best possible strengthening of the Main Conjecture.

## 2.6 Genesis of the Local Strengthening

The Local Strengthening did not arise in a hunt for a strongest possible conjecture. Rather, it arose naturally because it is sometimes easier to prove than the Main Conjecture. Here we explain why.

A class $\mathcal{G}$ of graphs is *hereditary* if for any graph $G \in \mathcal{G}$, every induced subgraph of $G$ is in $\mathcal{G}$. One way to prove the Main Conjecture for such a class of graphs is to prove that if $G \in \mathcal{G}$, then $G$ contains a stable set $S$ for which $\gamma(G - S) < \gamma(G)$. Since $S$ can be a colour class in a proper colouring of $G$, $\chi(G - S) \geq \chi(G) - 1$. Thus we can prove the Main Conjecture for $\mathcal{G}$ by induction. This approach involves finding an $S$ whose removal either lowers $\omega(G)$, or lowers $\Delta(G)$ by two.

(Any stable set $S$ in a graph can be extended to a maximal stable set whose removal lowers $\Delta(G)$ by at least one.) However, finding such an $S$ is not always possible.

To make this inductive approach easier, we introduced the invariant $\omega(v)$ for every vertex $v$; recall that it is the size of the largest clique containing $v$. Now we ask: can we find a stable set $S$ whose removal lowers $d(v) + \omega(v)$ by two for every vertex $v$ maximizing $d(v) + \omega(v)$? In other words, can we rip out a stable set and reduce $\gamma_l$, rather than trying to reduce $\gamma$?

This idea arose during efforts to prove the Main Conjecture for *antiprismatic thickenings*, which are an important subclass of claw-free graphs (we will define them in Chapter 9). If $G$ is an antiprismatic thickening with $\alpha(G) > 2$, it is very easy to remove a stable set and lower $\gamma_l(G)$, but it seems much more difficult to do the same for $\gamma(G)$. Thus to prove the Local Strengthening for antiprismatic thickenings we can repeatedly remove a stable set, lowering $\gamma_l(G)$ each time, until we are left with a graph with $\alpha \leq 2$. We then appeal to Theorem 2.15 to complete the proof.

We need more definitions and background before we give a formal proof of the Local Strengthening for antiprismatic thickenings in Chapter 10, but it is important to understand that the Local Strengthening is particularly appealing because it is often much easier to prove than the Main Conjecture. We will exploit this fact repeatedly when we prove the Main Conjecture for claw-free graphs.

### 2.6.1 Further tightness of the Local Strengthening

The Local Strengthening replaces the global invariant $\gamma(G)$ with the maximum over all local invariants $\gamma_l(v)$. Can we go one step further and formulate a conjecture that imposes a restriction for each local invariant simultaneously? We close the chapter by suggesting a further strengthening of the Main Conjecture in this vein and proving that it is not true.

**Proposition 1.** *For any graph $G$ there is a proper $\gamma_l(G)$-colouring of $G$ such that for any vertex $v$ of $G$, at most $\lceil \frac{1}{2}(d(v) + 1 + \omega(v))\rceil$ colours appear in the closed neighbourhood of $v$.*

In a proper $\gamma_l(G)$ colouring of a graph $G$, there is some vertex $v$ such that at most $\lceil \frac{1}{2}(d(v)+1+\omega(v))\rceil$ colours appear in the closed neighbourhood of $v$. The proposition insists that this property is satisfied by every vertex. The following theorem disproves the proposition.

**Theorem 2.17.** *For any positive integer $k$, there is a graph $G_k$ such that for any proper colouring of $G$, there is a vertex $v$ of degree $k$ whose neighbourhood of $v$ induces a $k$-coloured stable set.*

*Proof.* We will construct $G_k$ beginning with stable sets $V_1, V_2, \ldots, V_k, X$ that partition $V(G)$. Let $V_1$ be a stable set of size $N = k^3$; $N$ will define the sizes of the other stable sets. We call a vertex set a *transversal* of $V_1, \ldots, V_i$ if it has size $i$ and intersects each of $V_1, \ldots, V_i$ in exactly one vertex. If it is a stable set, we call it an *independent transversal*.

For $i > 1$, let $V_i$ be a stable set containing exactly one vertex for every transversal of $V_1, \ldots, V_{i-1}$. Accordingly, for every transversal $S$ of $V_1, \ldots, V_{i-1}$ there is a vertex of $V_i$ whose neighbourhood in $G_k[\cup_{j=1}^{i} V_j]$ is $S$. Finally, let $X$ be a stable set consisting of one vertex for each stable set of size $k$ in $G[\cup_{i=1}^{k} V_i]$; for every stable set $S$ of size $k$ in $G_k[\cup_{i=1}^{k} V_i]$ there is a unique vertex in $X$ with neighbourhood $S$.

We say that a colouring of $G_k$ is *good* if it is proper and there is no vertex in $X$ that sees $k$ different colours. It suffices to prove that there is no good colouring of $G$. Suppose we have a good colouring of $G_k$. First observe that each $V_i$ (moreso, every subset of it) receives fewer than $k$

colours. We will show that this forces a $k$-coloured independent transversal, giving us the desired contradiction.

There is some colour $c_1$ seen on at least $N/k$ vertices of $V_1$. Accordingly there are at least $N/k$ vertices of $V_2$ that cannot receive colour $c_1$, hence there is some colour $c_2$ seen on at least $(N/k)(1/k) = N/k^2$ vertices of $V_2$. Moving on to $V_3$, there are at least $(N/k)(N/k^2)$ vertices that cannot receive $c_1$ or $c_2$, so there is some third colour $c_3$ seen on at least $(N/k)(N/k^2)(1/k) = N^2/k^4$ vertices in $V_3$.

Continuing this analysis, we find that there are unique colours $c_1, c_2, \ldots, c_k$ such that for $i = 2, 3, \ldots, k$, the colour $c_i$ appears on at least $(N/k^2)^{2^{i-2}}$ vertices of $V_i$, and $c_1$ appears on at least $N/k$ vertices of $V_1$. In particular, for $i = 1, 2, \ldots, k$ there are at least $k$ vertices of colour $c_i$ in $V_i$.

We will find a $k$-coloured independent transversal $S$ of $V_1, \ldots, V_k$ starting at $V_k$ and working our way back as follows: From $V_i$, add to $S$ any vertex with colour $c_i$ with no neighbours in $S$. This is always possible, because at any point each vertex of $S$ will have at most one neighbour in $V_i$. Thus we can find our $k$-coloured independent transversal $S$, proving that the colouring is not good since there is a vertex in $X$ with neighbourhood $S$. This completes the proof. □

Notice that the theorem says nothing about the total number of colours used in the colouring; it is easy to see that $\chi(G_k) \leq k + 1$. The proposition may be true for claw-free graphs. The counterexample we just constructed contains a claw for $k \geq 2$. We conjecture that the fractional weakening of the proposition is also false.

**Conjecture 2.18.** *There exists a graph $G$ such that in any fractional colouring there is a vertex $v$ for which the stable sets intersecting $N(v)$ have total weight greater than $\frac{1}{2}(d(v) - 1 + \omega(v))$.*

We actually suspect that the graph constructed in the proof of Theorem 2.17 may also prove Conjecture 2.18.

# Part II

# Fractional and Integer Colourings

Over the next two chapters we explore the connection between fractional and integer colourings of graphs. The next chapter focuses on perfect graphs, for which the fractional and integer chromatic numbers are equal. As we will explain, the relationship between fractional and integer colourings of perfect graphs is actually much deeper than this. Perfect graphs have been the subject of much attention over the past four decades. We trace the development of the field and in doing so gather many useful tools that will help us deal with claw-free graphs in later chapters. The most useful of these are structural decompositions which arose in both the study of restricted subclasses of perfect graphs and the pursuit of the Strong Perfect Graph Conjecture.

In Chapter 4 we investigate two important classes of claw-free graphs for which the fractional and integer chromatic numbers are close together: line graphs and circular interval graphs. As we will see in Chapters 5, 8, and 9, these two classes suggest two composition operations that we will use to construct general claw-free graphs, and they are also used as base classes of these compositions. In a step towards proving the Main Conjecture for claw-free graphs, we will prove the Main Conjecture for line graphs and circular interval graphs. Our proof leads to a polynomial-time $\gamma$-colouring algorithm for these graphs.

# Chapter 3

# Fractional Colouring and Perfect Graphs

Since we are interested in graphs for which $\chi(G)$ is closer to $\omega(G)$ than to $\Delta(G)+1$, it is natural to consider graphs for which $\chi(G) = \omega(G)$. However, this property does not tell us anything about the structure of the graph other than the fact that it contains a $\chi(G)$-clique. For example, taking the disjoint union of any graph $G$ and a clique of size $\chi(G)$ gives us a graph for which $\chi(G) = \omega(G)$.

In this chapter we focus on graphs with the stronger property that $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. These widely studied *perfect graphs* were introduced by Claude Berge [Ber61], who was motivated by a problem of Shannon about the rate at which information can be transmitted across an empty channel without any chance of error [Sha56].

Early interest in perfect graphs was motivated by two conjectures due to Berge. The Weak Perfect Graph Conjecture (WPGC) proposes that $G$ is perfect precisely if $\overline{G}$ is perfect. The Strong Perfect Graph Conjecture (SPGC) proposes that $G$ is perfect precisely if neither $G$ nor $\overline{G}$ contains an odd hole. If neither $G$ nor $\overline{G}$ contains an odd hole we say that $G$ is *Berge*. The strong conjecture obviously implies the weak conjecture.

Since the fractional chromatic number lies between the clique number and the chromatic number, we know that $\chi_f(G) = \chi(G)$ for any perfect graph $G$. As we discuss later in this chapter, to actually find an optimal colouring we use the stronger fact that when $G$ is perfect the fractional stable set polytope $QSTAB(G)$ has only integer extreme points, and is therefore equal to the stable set polytope $STAB(G)$; we define these polytopes in the next section. It turns out that in fact the stable set polytopes of perfect graphs can be used to characterize exactly those polytopes of the form $Ax = 1$ which have only integer extreme points and for which $A$ is a $(0,1)$ matrix. For any such $A$, we can solve every integer program of the form

$$\max \ cx \ \text{ s.t. } Ax = 1 \text{ and } x \text{ is integral}$$

by solving the linear program obtained by dropping the integrality constraint. This fact generated considerable interest in perfect graphs amongst combinatorial optimizers.

This second, polyhedral motivation is of more direct interest to us, since we will use the facts that $\chi(G)$ is equal to or near $\chi(G)$ and that $QSTAB(G) = STAB(G)$ for various classes of graphs repeatedly when studying claw-free graphs. So we begin the chapter with a discussion of the ratio between $\chi_f(G)$ and $\chi(G)$ for general graphs, a description of the proof that $QSTAB(G) = STAB(G)$ for perfect graphs, and a discussion of algorithms for computing $\chi(G)$ and solving other optimization problems on perfect graphs.

We also discuss the proofs of Berge's two conjectures, which are separated by thirty years. Lovász' 1972 proof of the WPGC is short and elegant, relying on the *Replication Lemma* and homogeneous cliques. We will need to consider homogeneous cliques throughout our work on claw-free graphs in later chapters. In contrast to the proof of the WPGC, the proof of the SPGC due to Chudnovsky, Robertson, Seymour, and Thomas is over 100 pages long and we will not be able to reproduce it. We will however sketch the proof for two reasons. First, it implies the Main Conjecture for Berge graphs. Second, we will introduce structural decompositions and tools that will be of great use to us in our study of claw-free graphs. In particular, we rely heavily on *2-joins* and *homogeneous pairs* for reasons we explain fully in Chapters 5 and 6 respectively.

To round off the chapter, we discuss recognition algorithms for perfect graphs.

## 3.1 Fractional colouring

In Section 1.2 we defined the fractional chromatic number $\chi_f(G)$ of a graph $G$ as the optimal value of the following linear program (1.2):

$$\begin{array}{rl} \text{minimize} & \displaystyle\sum_{S\in\mathcal{S}(G)} w_S \\ \text{subject to} & \forall v \in V(G), \displaystyle\sum_{S\ni v} w_S \geq 1 \\ \text{and} & \forall S \in \mathcal{S}(G), w_S \in [0,1] \end{array}$$

The feasible region of this program is $STAB(G)$, the *stable set polytope* of $G$. As we will show, $\chi_f(G)$ is also equal to the minimum $\beta$ such that $(\frac{1}{\beta}, \frac{1}{\beta}, \ldots, \frac{1}{\beta})$ is in $STAB(G)$ – our result from the previous chapter implies that $(\frac{1}{\gamma_l}, \frac{1}{\gamma_l}, \ldots, \frac{1}{\gamma_l})$ is in $STAB(G)$. If we can optimize over the stable set polytope of $G$ efficiently, we can determine the fractional chromatic number of $G$ efficiently.

We consider another polytope, the *fractional stable set polytope* of $G$, denoted by $QSTAB(G)$. It consists of all nonnegative vertex weightings on $G$ such that no stable set of $G$ has total weight more than 1. For all graphs, $STAB(G) \subseteq QSTAB(G)$. Later in this chapter we will show that a graph is perfect precisely if $STAB(G) = QSTAB(G)$ and furthermore, we can optimize over these polytopes efficiently.

For a probability distribution $p$ on the stable sets of a graph, the value $\mathbf{Pr}(v \in S)$ is called the *marginal* of $p$ at $v$. We now present the relationship between $STAB(G)$, the fractional chromatic number, and marginals achievable by probability distributions on the stable sets of a graph, which was hinted at in the proof of Theorem 2.10.

**Observation 3.1.** *The following statements are equivalent:*

*(1)* $\chi_f(G) = k$

*(2)* $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}) \in STAB(G)$

*(3) There is a probability distribution $p$ on the stable sets of $G$ such that the marginal of $p$ at each vertex is $\frac{1}{k}$.*

Thus the stable set polytope actually characterizes precisely (and in turn is defined by) the marginal vectors that can be achieved by some such $p$.

*Proof.* Suppose for some $k \in \mathbb{R}^+$ the $|V|$-dimensional vector $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$ is in the stable set polytope of $G$. Then there is a convex combination of stable sets of $G$ in which every vertex has weight $\frac{1}{k}$. Taking $k$ times this convex combination gives a combination of stable sets with total weight $k$ in which every vertex has weight 1. This is a fractional $k$-colouring of $G$, so $\chi_f(G) \leq k$ and so (2) implies (1). It is easy to see that the reverse implication is also true, i.e. that $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}) \in STAB(G)$ precisely if $\chi_f(G) \leq k$. Furthermore the convex combination of stable sets actually gives a probability distribution on stable sets of $G$ such that if $S$ is drawn from the distribution, then for any $v \in V$, $\mathbf{Pr}(v \in S) = \frac{1}{k}$. Specifically, the probability of $S$ in the distribution is precisely the weight of $S$ in the convex combination. So (2) implies (3) and again it is clear that the converse is also true. $\square$

We mentioned earlier that any graph $G$ satisfies $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$. This result is due to Lovász, who gave a different proof of a more general result on set covers [Lov75]. The proof we present uses the relationship between $\chi_f(G)$ and probability distributions on the stable sets of $G$.

**Theorem 3.2.** *For any graph $G$, $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$.*

*Proof.* By Observation 3.1 there is some probability distribution on the stable sets of $G$ such that if $S$ is drawn from the distribution, for all $v \in V$ we have $\mathbf{Pr}(v \in S) = \frac{1}{\chi_f(G)}$. We colour $G$ by drawing $\lceil \log n \cdot \chi_f(G) \rceil$ random colour classes from this distribution and arguing that with nonzero probability, each vertex is in at least one colour class. This gives a $\lceil \log n \cdot \chi_f(G) \rceil$-colouring of $G$, since if a vertex is in more than one colour class we can just choose one of these colours arbitrarily for the vertex.

Let $v$ be any fixed vertex. Every colour class $S$ has the property that $\mathbf{Pr}(v \in S) = \frac{1}{\chi_f(G)}$. Since the colour classes are drawn independently from the same distribution, the probability that $v$ is in no colour class is

$$\left(1 - \frac{1}{\chi_f(G)}\right)^{\lceil \log n \cdot \chi_f(G) \rceil} \leq \left(\left(1 - \frac{1}{\chi_f(G)}\right)^{\chi_f(G)}\right)^{\frac{\lceil \log n \cdot \chi_f(G) \rceil}{\chi_f(G)}} < \left(\frac{1}{e}\right)^{\log n} = \frac{1}{n}.$$

Thus the probability of every vertex being in at least one colour class is greater than $1 - n\frac{1}{n} = 0$. The theorem follows. $\square$

This proof is an example of the simple but powerful *first moment method* (see, for example, Chapter 3 in [MR00]).

In Chapter 1 we showed that $\omega(G) \leq \chi_f(G) \leq \chi(G)$. Here we give examples that prove $\chi_f(G)$ can be at either end of this range, even if the difference between $\omega(G)$ and $\chi(G)$ is large. This is in contrast to the previous chapter, in which we disproved the analogous statement for $\chi(G)$ and the range $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. The fractional chromatic number can also be in the middle of the range $[\omega(G), \chi(G)]$.

We first show that $\chi_f(G)$ can be equal to $\omega(G)$ but far from $\chi(G)$. Given positive integers $a$ and $b$ for which $b \leq \frac{a}{2}$, the *Kneser graph* $K_{a:b}$ has order $\binom{a}{b}$. Its vertices correspond to the $b$-subsets of a base set of size $a$, and two vertices are adjacent precisely if their corresponding subsets are disjoint. (The famous Petersen graph is actually $K_{5:2}$.) For any integers $k \geq 2$ and $b \geq 2$, the graph $K_{kb:b}$ has clique number $k$. One can use the well-known Erdős-Ko-Rado Theorem [EKR61] and the vertex-transitive property of Kneser graphs to prove that $\chi_f(K_{kb:b}) = k$; the proof is given in

Chapter 3 of [SU97]. Lovász [Lov78] proved that $\chi(K_{a:b}) = a-2b+2$, and so $\chi(K_{kb:b}) = (k-2)b+2$, far away from the fractional chromatic number.

To see that $\chi_f(G)$ and $\chi(G)$ can be equal and far from $\omega(G)$, let $G$ be the line graph of $C_5$ with $k$ copies of every edge (as we did in Section 2.1). If $k$ is even, $G$ has clique number $2k$ and independence number 2, thus $\chi_f(G) \geq \frac{5}{2}k$. It is easy to show that $\chi(G) = \frac{5}{2}k$, so $\chi_f(G) = \chi(G)$.

To see that $\chi_f(G)$ can be in the middle of the range $[\omega(G), \chi(G)]$, let $G$ be $k$ disjoint copies of $C_5$ with all possible edges between them. Then $\omega(G) = 2k$, $\chi_f(G) = \frac{5}{2}k$, and $\chi(G) = 3k$. We will revisit this example later in the thesis. Larsen, Propp, and Ullman [LPU95] proved that Mycielski's sequence of graphs $G_k$, mentioned in Section 2.1 (see [Myc55]), have $\omega(G_k) = 2$, $\chi(G_k) = k$, and $\chi_f(G_k) = \chi_f(G_{k-1}) + \frac{1}{\chi_f(G_{k-1})}$. Thus $\chi_f(G_k)$ asymptotically approaches $\sqrt{2k-2}$.

### 3.1.1 Inapproximability of $\chi$ and $\chi_f$

We have shown that $\omega(G)$, $\chi_f(G)$, and $\chi(G)$ can be far apart from one another, even though they are all equal for perfect graphs. It turns out that all three of them are difficult to compute and even approximate in general. Since the fractional chromatic number can be found by solving a linear program, one might make the mistake of thinking that we can find $\chi_f(G)$ in polynomial time. But the linear program contains a variable for every stable set of $G$, and can therefore have exponential size compared to the size of $G$.

Indeed it is *NP*-hard[1] to approximate the chromatic number of a graph to within a factor of $|V|^{\epsilon_\chi}$ for some positive $\epsilon_\chi$. Bellare, Goldreich, and Sudan proved that this $\epsilon_\chi$ is at least $\frac{1}{7}$ [BGS98]. By Theorem 3.2, $\chi(G)$ exceeds $\chi_f(G)$ by at most a factor of $\log(|V|)$, thus it is hard to compute the fractional chromatic number of a graph in general under the assumption that $P \neq NP$ because $\chi_f(G)$ gives a $\log(|V|)$-approximation to $\chi(G)$. As pointed out in Exercise 10.8 in [AL95], this implies that approximating $\chi_f(G)$ is also hard.

By results of Håstad [Hås99] and Feige and Killian [FK98], if we make stronger assumptions[2] than $P \neq NP$, then it is hard to approximate $\omega$, $\chi$, and $\chi_f$ to within a factor of $|V|^{1-\epsilon}$ for any $\epsilon > 0$.

In spite of these difficulties it is possible to efficiently compute the chromatic number for a perfect graph and even find an optimal colouring, as we will show later in this chapter. First we must lay out the foundation of these optimization results: the Replication Lemma and the proof of the Weak Perfect Graph Theorem, which have extensive implications in both structural and polyhedral aspects of perfect graph theory.

## 3.2 The Replication Lemma and the Weak Perfect Graph Theorem

In 1972, Lovász proved the Weak Perfect Graph Conjecture [Lov72]. His proof hinges on the following lemma.

**Definition 3.3.** *We* replicate *a vertex $v$ in $G$ by adding a vertex $v'$ to $G$ with neighbourhood $\bar{N}(v)$ (recall that $\bar{N}(v)$ denotes the closed neighbourhood $\{v\} \cup N(v)$ of a vertex $v$). In other words, we*

[1]See [GJ79] for definitions related to *NP*-hardness.

[2]Inapproximability of $\omega$ relies on the assumption that $NP \neq co\text{-}RP$, and inapproximability of $\chi$ and $\chi_f$ relies on the assumption that $NP \neq ZPP$. We refer the interested reader to [Pap94] for the definitions of these complexity classes. We know that $P \subseteq ZPP \subseteq co\text{-}RP \subseteq NP$ but all four classes may be equal, although this is widely disbelieved.

*add a* twin *of $v$ to $G$.*

**Replication Lemma** (Lovász[3] [Lov72])**.** *If $G$ is obtained from a perfect graph by replicating a vertex then $G$ is perfect.*

*Proof.* We need to show that if $G'$ is obtained by adding a twin $v'$ of a vertex $v$ to a perfect graph $G$ then for every induced subgraph $H'$ of $G'$, we have $\chi(H') = \omega(H')$.

Since $G$ is perfect we are done if $v' \notin H'$ because in this case $H'$ is an induced subgraph of $G$. Similarly, we are done if $v' \in H'$ but $v \notin H'$ since in this case $H'$ is isomorphic to $H' + v - v'$

So assume both $v$ and $v'$ are in $H'$ and let $H = H' - v'$. If $v$ is in an $\omega(H)$ clique in $H$ then $\omega(H') = \omega(H) + 1$ and we can $\omega(H')$-colour $H'$ using an $\omega(H)$-colouring of $H$ and a new colour for $v'$. If $v$ is in no $\omega(H)$ clique, we $\omega(H)$-colour $H$ and let the stable set $S$ be the colour class containing $H$. Observe that $\omega(H - S) = \omega(H) - 1$. It follows that $H - (S - v)$ has no $\omega(H)$ clique and since $G$ is perfect, $\chi(H - (S - v)) = \omega(H) - 1$. Using $S - v + v'$ as a final colour class gives us the desired $\omega(H')$-colouring of $H'$. □

We immediately get the following corollary.

**Corollary 3.4.** *If $G$ is obtained from a perfect graph through a sequence of vertex replications then $G$ is perfect.*

This motivates an important definition.

**Definition 3.5.** *To* substitute *a graph $H$ for a vertex $v$ in a graph $G$, we first take the disjoint union of $G - v$ and $H$, then make each vertex in $H$ adjacent to each vertex in $N(v)$.*

So repeatedly replicating a vertex is equivalent to substituting a clique for the vertex, and this operation preserves perfection. Lovász actually proved that if a perfect graph $H$ is substituted for a vertex $v$ in a perfect graph $G$, the resulting graph is perfect (see Chapter 2 in [RR01]).

The Replication Lemma is also the key to proving that $QSTAB(G) = STAB(G)$ if $G$ is perfect, as we will show in the next section. We say that a set $S$ of vertices in $G$ is a *homogeneous set* if $|S| \geq 2$, $|V(G) \setminus S| \geq 2$, and every vertex in $V(G) \setminus S$ sees either all or none of $S$. If $S$ is a clique, we say that $S$ is a *homogeneous clique*. The notions of replication, substitution, and homogeneous sets are closely related: If we replicate a vertex in a graph on at least three vertices, the resulting graph will contain a homogeneous clique. If we substitute $H$ for $v$ in $G$, then the vertices of $H$ will be a homogeneous set in the resulting graph.

This idea remains extremely important when we shift our focus from perfect graphs to claw-free graphs in the next chapter. In particular, we will rely heavily on the notion of *thickenings*, which generalize repeated vertex replication, and *homogeneous pairs of cliques*, which generalize homogeneous cliques.

We close this section with a proof the Weak Perfect Graph Conjecture, which is now the Weak Perfect Graph Theorem.

**Theorem 3.6** (Lovász [Lov72])**.** *If $G$ is perfect then $\overline{G}$ is perfect.*

[3]Fulkerson had previously noted that the Replication Lemma would be enough to prove the Weak Perfect Graph Conjecture [Ful72].

*Proof.* It suffices to prove that if $G$ is perfect, then $\chi(\overline{G}) = \omega(\overline{G})$, so assume $G$ is a minimal counterexample to this statement. Thus every proper induced subgraph $H$ of $G$ can be covered with $\alpha(H)$ cliques.

Let $t$ be the number of maximum stable sets in $G$, and for every vertex $v$ of $G$ let $t_v$ be the number of maximum stable sets containing $v$. Construct the graph $G'$ from $G$ by deleting every vertex $v$ for which $t_v = 0$, then replicating every remaining vertex $t_v - 1$ times. Thus we get $G'$ from $G$ by substituting a clique $C_v$ of size $t_v$ for every vertex $v$. We can label each vertex of $C_v$ with a unique maximum stable set of $G$ containing $v$; it follows that $G'$ is $t$-colourable.

$G'$ has $t\alpha(G)$ vertices and $\alpha(G') = \alpha(G)$, so $\chi(G') \geq |V(G')|/\alpha(G') = t$, so $\chi(G') = t$. By Corollary 3.4, $G'$ is perfect and therefore contains a clique $C'$ of size $t$. The maximum stable sets of $G$ give a $t$-colouring of $G'$, so $C'$ intersects each of these stable sets. Let $C \subseteq V(G)$ be the set of vertices $v$ for which $C'$ intersects $C_v$; $C$ is clearly a clique and it intersects every maximum stable set in $G$, so $\alpha(G - C) = \alpha(G) - 1$. By minimality of $G$, $G - C$ can be covered by $\alpha(G) - 1$ cliques. It follows that $G$ can be covered by $\alpha(G)$ cliques, so $\chi(\overline{G}) = \omega(\overline{G})$, a contradiction. □

**Remark:** In Section 2.2 we noted that if every vertex in a graph $G$ is contained in the same number of maximum stable sets, then $\chi_f(G) = |V(G)|/\alpha(G)$, and we get an optimal fractional colouring by giving equal weight to every maximum stable set. The above proof follows this line of thinking, as the $t$-colouring of $G'$ is the result of fractionally colouring $G$ using only maximum stable sets so that every vertex $v$ receives total weight $t_v$.

To prove that $QSTAB(G) = STAB(G)$ when $G$ is perfect, we will use a similar construction based on repeated replication. But instead of substituting a clique for each vertex we will substitute a stable set for each vertex, noting that the resulting graph is still perfect.

## 3.3 Optimizing over perfect graphs

We have established that finding the chromatic number or fractional chromatic number of a graph is hard in general. In this section we will explain how to optimally colour a given perfect graph $G$ in polynomial time. Since $G$ is perfect, computing $\chi(G)$ is equivalent to computing $\chi_f(G)$ and $\omega(G)$. By the Weak Perfect Graph Theorem, if we can compute $\chi(G)$ efficiently we can also compute $\chi(\overline{G})$ and $\alpha(G)$ efficiently. Even more impressive is the fact that we can actually find a maximum stable set and optimally colour $G$ in polynomial time.

Our first step is to calculate $\omega(G)$, which we do by optimizing over $STAB(\overline{G})$ to find a maximum stable set of $\overline{G}$. Grötschel, Lovász, and Schrijver proved that we can approximately solve a convex optimization problem efficiently if we can solve the associated separation problem approximately. For integer programs in which the vertices of the feasible region are $(0, 1)$ vectors, this implies that we can find an exact solution efficiently if we can separate efficiently. We refer the reader to [GLS93] for further explanation.

To optimize over perfect graphs we exploit the equality of the stable set polytope and fractional stable set polytope. This equality was first proved using antiblocking polyhedra [Ful72]. Chvátal [Chv75] later gave a simple proof that $QSTAB(G) = STAB(G)$ for perfect graphs, based on Lovász' proof of the Weak Perfect Graph Theorem.

**Theorem 3.7.** *If $G$ is perfect, then $QSTAB(G) = STAB(G)$.*

*Proof.* We already know that $STAB(G) \subseteq QSTAB(G)$. To show that $QSTAB(G) \subseteq STAB(G)$ it suffices to show that there is no direction in which $QSTAB(G)$ exceeds $STAB(G)$. Thus since the vertices of both polyhedra are rational and in the nonnegative orthant, it suffices to show that there is no rational vector contained in $QSTAB(G) \setminus STAB(G)$. That is for any rational vector $c$, the maximum of $c \cdot x$ over $QSTAB(G)$ is achieved by a point in $STAB(G)$, i.e. a characteristic vector of a stable set in $G$. If $c$ is the 1-vector, this follows from the fact that $G$ can be covered by $\alpha(G)$ cliques by the WPGT, so the maximum of $c \cdot x$ over $QSTAB(G)$ is $\alpha(G)$. We need only consider nonnegative integer-valued $c$ because we can rescale $c$ to have integer values without changing its direction.

For arbitrary nonnegative integer-valued $c$, we construct a graph $G_c$ by substituting a stable set of size $c_v$ for every vertex $v$ of $G$. This is equivalent to substituting a clique of size $c_v$ for every vertex $v$ of $\overline{G}$, then taking the complement. So by the Weak Perfect Graph Theorem and the Replication Lemma, $G_c$ is perfect. Every stable set $S$ in $G$ corresponds to a stable set of size $\sum_{v \in S} c_v$ in $G_c$. Furthermore, just as we bounded the chromatic number of $G'$ in the proof of the WPGT, we can see that $\alpha(G_c)$ is the maximum of $c \cdot x$ such that $x$ is the characteristic vector of a stable set in $G$.

Since $G_c$ is perfect it can be covered by $\alpha(G_c)$ cliques, so there is no point $y_c \in QSTAB(G_c)$ for which $1 \cdot y_c > \alpha(G_c)$. Suppose there is a point $y \in QSTAB(G)$ for which $c \cdot y > \alpha(G_c)$. This point corresponds to a weighted combination of stable sets $S_i$ in $G$, and each $S_i$ corresponds to a stable set of size $\sum_{v \in S_i} c_v$ in $G_c$. Combining these stable sets in $G_c$ gives us a point $y_c \in QSTAB(G_c)$ for which $1 \cdot y_c > \alpha(G_c)$, a contradiction. □

We mentioned earlier that $QSTAB(G) = STAB(G)$ precisely if $G_c$ is perfect. If $G$ is not perfect then it contains an odd hole or antihole as an induced subgraph by the Strong Perfect Graph Theorem. Knowing this, it is a simple matter to prove that $QSTAB(G) \neq STAB(G)$.

The key to optimizing over perfect graphs is a convex body called the *theta body* of $G$, denoted $TH(G)$. The theta body, first introduced by Lovász [Lov79], lies between $QSTAB(G)$ and $STAB(G)$ for any graph, so for any perfect graph $G$ the three bodies are equal. Grötschel, Lovász, and Schrijver proved that we can approximately solve convex optimization problems over the theta body efficiently using the *ellipsoid method*, which is a form of geometric binary search [GLS81]. Since the vertices of $TH(G)$ are integral, we can find a maximum stable set in a perfect graph in polynomial time. In fact we can find a maximum weight stable set for any nonnegative integer weighting on the vertices of $G$.

Since $\overline{G}$ is also perfect by the WPGT, we can also efficiently compute $\omega(G)$, $\chi(G)$ and $\chi_f(G)$. Recall that $\chi_f(G) \leq k$ precisely if $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$ is in $STAB(G)$, so the vector $(\frac{1}{\omega}, \frac{1}{\omega}, \ldots, \frac{1}{\omega})$ is in $STAB(G)$ and it represents a convex combination of stable sets. Equivalently, it can be expressed as a convex combination of vertices of $STAB(G)$. We can find such an expression efficiently [GLS93]; we now show how this fact allows us to find an optimal colouring of $G$.

If a stable set $S$ is given nonzero weight in a fractional $\omega(G)$-colouring of a perfect graph $G$, then $S$ intersects every maximum clique of $G$. Therefore $\omega(G - S) = \omega(G) - 1$ and by perfection, $\chi(G - S) = \chi(G) - 1$. Thus we can use $S$ as a colour class in an optimal colouring of $G$ and recursively colour $G - S$ using $\chi(G) - 1$ colours. So to efficiently find an optimal colouring of a perfect graph, it suffices to find such an $S$ efficiently. To do this, we first compute $\omega(G)$ by optimizing over the theta body of $\overline{G}$. We then express $(\frac{1}{\omega}, \frac{1}{\omega}, \ldots, \frac{1}{\omega})$ as a convex combination of vertices of $STAB(G)$, and take $S$ to be the stable set represented by some vector given nonzero weight in this expression.

## 3.4 Combinatorial algorithms and some subclasses of perfect graphs

The decades leading up to the proof of the Strong Perfect Graph Theorem saw a huge amount of work in the theory of perfect graphs. In particular, many classes of Berge graphs were shown to be perfect, and many properties of minimal imperfect graphs[4] were proved. In this section we discuss some classes of perfect graphs that yield efficient detection algorithms and combinatorial algorithms for solving optimization problems. These algorithms are of interest to us for two reasons. First, the ellipsoid method is not very efficient in practice, and optimizing over the theta body does not give us a strong intuition as to how we are exploiting the structure of the graph. Second, the tools and ideas we introduce in this section are of use to us in more general settings, in particular when dealing with claw-free graphs.

### 3.4.1 Bipartite graphs and their complements

A graph is bipartite precisely if it is 2-colourable. Bipartite graphs are perfect. It is easy to show that a graph is bipartite precisely if it contains no cycle of odd length.

If $G$ is the complement of a bipartite graph, then $G$ is *cobipartite*. Equivalently, $G$ is cobipartite if its vertices can be covered by two cliques. Clearly an optimal colouring of a cobipartite graph consists of a maximum matching in $\overline{G}$ and a set of singleton colour classes. By König's Theorem [Kön16], the maximum matching in a bipartite graph has size equal to that of a minimum vertex cover, which implies that cobipartite graphs are perfect without using the perfection of bipartite graphs. We will use the perfection of cobipartite graphs repeatedly throughout the remaining chapters, particularly in the context of homogeneous pairs of cliques, which we define later in this section.

### 3.4.2 Line graphs of bipartite graphs

The edges of a bipartite graph $H$ can be coloured using $\Delta(H)$ colours. This follows easily from the fact that a maximum matching and minimum cover have the same size in a bipartite graph. Since $\omega(L(H)) = \Delta(H)$ for bipartite $H$, this implies that the line graph of any bipartite graph is perfect. The Replication Lemma extends this result to line graphs of bipartite multigraphs, and the WPGT implies that the complement of the line graph of a bipartite graph is perfect. Line graphs of bipartite graphs make up an important class of claw-free Berge graphs, and as we show in the next section, they also make up an important class of general Berge graphs.

### 3.4.3 Clique cutsets

A clique $C$ in a graph $G$ is a *clique cutset* if $G - C$ is disconnected. If $G_1, \dots, G_l$ are the connected components of $G - C$, then we can construct $G$ from the graphs $G_i \cup C$ by identifying $C$ in each of the graphs. We call this operation by which we paste together the graphs $G_i \cup C$ a *clique sum*.

If we have a $k_i$-colouring of each graph $G_i \cup C$, then in each colouring, every vertex of $C$ receives a different colour. Thus by permuting the colour classes we can obtain colourings that agree on $C$; together these colourings yield a proper colouring of $G$ using $\max_{1 \leq i \leq l} k_i$ colours.

It follows that if $\mathcal{B}$ is a class of perfect graphs and $\mathcal{B}^C$ is a hereditary class of graphs each of which is in $\mathcal{B}$ or contains a clique cutset, then $\mathcal{B}^C$ is also a class of perfect graphs.

[4]An imperfect graph is *minimal imperfect* if all of its proper induced subgraphs are perfect.

Whitesides [Whi81] proved that we can find a clique cutset in $O(nm)$ time. Tarjan extended this [Tar85], proving that in $O(nm)$ time we can construct a *clique cutset tree* for any graph. A clique cutset tree of a graph $G$ is a rooted tree defined recursively as follows: the root of the tree corresponds to a clique cutset $C$ of $G$, or to $G$ if $G$ contains no clique cutset. From the root we hang the clique cutset trees of the resulting graphs $G_i \cup C$. Thus if we have a polynomial-time algorithm for colouring graphs in $\mathcal{B}$, we have a polynomial-time algorithm for colouring graphs in $\mathcal{B}^C$ that involves working our way up the clique cutset tree, starting with the leaves.

We now describe some classes of perfect graphs for which this approach is particularly effective.

#### Chordal graphs

A graph is *chordal* if it contains no holes. Chordal graphs, also known as triangulated or Gallai graphs, were first identified and characterized by Dirac [Dir61]. They are perfect, so we can colour them in polynomial time using the ellipsoid method. Dirac proved that $G$ is chordal if and only if every induced subgraph of $G$ is a clique or has a clique cutset. Thus if $\mathcal{B}$ is the class of cliques then $\mathcal{B}^C$ is the class of chordal graphs. Therefore we have an algorithm for colouring chordal graphs that is far more efficient and illuminating than the polyhedral approach we use for general perfect graphs.

Fulkerson and Gross proved that $G$ is chordal precisely if every induced subgraph of $G$ has a *simplicial vertex*, i.e. a vertex whose neighbourhood induces a clique [FG65]. Thus any chordal graph $G$ has a *simplicial ordering* of its vertices in which we can remove simplicial vertices one at a time until only a single vertex remains. Unless $G$ is a clique, the neighbourhood of any simplicial vertex is a clique cutset. We can find a simplicial vertex in a chordal graph in $O(m)$ time, which is faster than the best known algorithm for general graphs.

#### Clique-separable and $i$-triangulated graphs

Let $\mathcal{B}$ be the class of graphs that are complete multipartite or a clique joined to a bipartite graph. In this case $\mathcal{B}^C$ is the class of *clique-separable* graphs. For $G \in \mathcal{B}$, if some component of $\overline{G}$ is not a clique, then it is a cobipartite graph and all other components of $\overline{G}$ are isolated vertices. Thus $\overline{G}$ is perfect and it follows that $\mathcal{B}$ is a subclass of perfect graphs. The structure of $\overline{G}$ yields an efficient algorithm for colouring graphs in $\mathcal{B}$, so we immediately get an efficient combinatorial algorithm for colouring clique-separable graphs.

A graph is *$i$-triangulated* if every cycle of odd length has at least two noncrossing chords. Gallai [Gal62] proved that these graphs are perfect, and in fact they form a subset of clique-separable graphs. It turns out that we can use the structure of $i$-triangulated graphs to find a maximum-weight induced $k$-partite subgraph in polynomial time for any fixed $k$ [AKK$^+$08]. To do this, we decompose using clique cutsets with special properties which we can show exist because of the special structure of $i$-triangulated graphs.

### 3.4.4 Star cutsets, even pairs, and weakly triangulated graphs

A *star cutset* $A$ in a graph $G$ is a cutset such that some vertex $v$ in $A$ sees every other vertex in $A$. We call $v$ the *centre* of the star cutset. Note that every clique cutset is also a star cutset. Chvátal [Chv85] proved that no minimal imperfect graph contains a star cutset, and so by the WPGT its complement does not contain a star cutset.

**Lemma 3.8** (Star Cutset Lemma)**.** *No minimal imperfect graph contains a star cutset.*

*Proof.* Suppose $G$ is minimal imperfect and contains a star cutset $A$ with centre $v$, and let the components of $G-A$ be $G_i$ for $1 \leq i \leq l$. Then each $G_i \cup A$ is perfect; take an $\omega(G_i \cup A)$-colouring of each graph and permute the colour classes so that $v$ always receives the same colour. Let the stable set $S$ in $G$ be the union of these colour classes. Every graph $(G_i \cup A) - S$ is $\omega(G) - 1$ colourable and hence has no $\omega(G)$ clique, so $G - S$ has no $\omega(G)$ clique. Since $G - S$ is perfect it is $\omega(G) - 1$ colourable so $G$ is $\omega(G)$ colourable, a contradiction. □

Given a class $\mathcal{B}$ of perfect graphs, let $\mathcal{B}^*$ be the class of graphs that are in $\mathcal{B}$ or admit a star cutset or admit a star cutset in the complement. The Star Cutset Lemma implies that $\mathcal{B}^*$ is a class of perfect graphs. By properties of star cutsets pointed out by Chvátal [Chv85], it is easy to find a star cutset in polynomial time. However, unlike the case for clique cutsets, we do not necessarily have an efficient combinatorial algorithm for colouring graphs in $\mathcal{B}^*$ since we do not know how to paste together the colourings of $G_i \cup A$ given a star cutset $A$.

In certain cases, however, we can further exploit the structure of $\mathcal{B}^*$ to colour graphs efficiently. A graph is *weakly triangulated* if it contains no hole or antihole of length greater than four. The class of weakly triangulated graphs lies between Berge graphs and chordal graphs. Hayward [Hay85] proved that weakly triangulated graphs are the class $\mathcal{B}^*$ when $\mathcal{B}$ is the class of graphs on at most two vertices, so they are perfect. To find a combinatorial colouring algorithm for weakly triangulated graphs we must introduce the notion of *even pairs*.

Two nonadjacent vertices $u$ and $v$ in a graph form an *even pair* if every chordless path between $u$ and $v$ has an even number of edges. For an even pair $\{u, v\}$ in a graph $G$ we contract $\{u, v\}$ by replacing $u$ and $v$ with a new vertex $uv$ adjacent to $N(u) \cup N(v)$. We use $G/uv$ to denote the resulting graph.

Even pairs are of interest to us because contracting on an even pair in a perfect graph always results in a perfect graph, as proved by Fonlupt and Uhry [FU82]. Their result follows from the fact that neither the chromatic number nor the clique number changes:

**Theorem 3.9.** *Let $\{u, v\}$ be an even pair in a graph $G$. Then $\omega(G/uv) = \omega(G)$ and $\chi(G/uv) = \chi(G)$.*

*Proof.* Obviously $\omega(G/uv) \geq \omega(G)$. Suppose $\omega(G/uv) > \omega(G)$. Then there is a maximum clique in $G$ containing vertices $w$ adjacent to $v$ but not $u$, and $x$ adjacent to $u$ but not $v$. The four vertices $v, w, x, u$ induce a path on three vertices, contradicting the fact that $\{u, v\}$ is an even pair.

Given a $\chi(G/uv)$ colouring of $G/uv$ we get a $\chi(G/uv)$ colouring of $G$ by giving both $u$ and $v$ the colour appearing on $uv$, so $\chi(G) \leq \chi(G/uv)$.

If there is a $\chi(G)$ colouring of $G$ such that $u$ and $v$ receive the same colour, we can $\chi(G)$ colour $G/uv$ by giving $uv$ that same colour. Suppose we have a $\chi(G)$ colouring of $G$ such that $u$ and $v$ receive different colours, and let $G'$ be the subgraph of $G$ induced on these two colour classes. If $u$ and $v$ are in the same component of $G'$ then we get an odd chordless path from $u$ to $v$, contradicting the fact that $\{u, v\}$ is an even pair. Therefore we can take the component of $G'$ containing $v$ and exchange the two colour classes to reach a proper $\chi(G)$ colouring of $G$ in which $u$ and $v$ receive the same colour. It follows that $\chi(G) = \chi(G/uv)$. □

This idea of reducing a graph without changing its chromatic number will be of use to us when we colour claw-free graphs, particularly in the setting of homogeneous pairs of cliques. Note, however, that contracting on an even pair can increase the maximum degree of the graph, so even pairs are of little use to us when considering the Main Conjecture and the Local Strengthening.

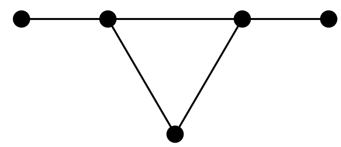

Figure 3.1: A bull.

A graph $G$ is called *even-contractile* if we can reduce $G$ to a clique through a sequence of even pair contractions. If the resulting sequence of graphs is $G = G_0, G_1, \ldots, G_k$ where $G_k$ is a clique of size $\chi(G)$, we can optimally colour $G$ as follows. Start with a colouring of $G_k$. Having optimally coloured $G_i$ for some $i$, and letting $\{u_i, v_i\}$ be the even pair in $G_{i-1}$ for which $G_i = G_{i-1}/u_iv_i$, we $\chi(G)$-colour $G_{i-1}$ by giving both $u_i$ and $v_i$ the colour appearing on $u_iv_i$ in the colouring of $G_i$, and using the same colouring on $G_{i-1} - u_i - v_i$ and $G_i - u_iv_i$.

This approach leads to a polynomial-time colouring algorithm for even-contractile graphs as long as we can find the desired sequence of even pairs in polynomial time. However, the problem of deciding whether or not $G$ admits an even pair is *co-NP*-complete (and thus not believed to be solvable in polynomial time) in general [Bie91].

Hayward, Hoáng, and Maffray proved that every weakly triangulated graph is even-contractile [HHM89]. Since contracting on an even pair in a weakly triangulated graph does not always leave a weakly triangulated graph, they considered a special type of even pairs. A 2*-pair* is an even pair $\{u, v\}$ in which every chordless path from $u$ to $v$ has length two. Hayward, Hoáng, and Maffray proved that every weakly triangulated graph is either a clique or contains a 2-pair. Contracting a 2-pair will not create a long hole or antihole, so this easily implies that every weakly triangulated graph is even-contractile. To colour weakly triangulated graphs efficiently it is enough to know that we can find a 2-pair in polynomial time. This is easy, because $\{u, v\}$ is a 2-pair precisely if $u$ and $v$ are in different components of $G - (N(u) \cap N(v))$. Arikati and Pandu Rangan later gave an $O(nm)$-algorithm for finding a 2-pair in an arbitrary graph if one exists [AP91].

### 3.4.5 Bull-free Berge graphs and homogeneous pairs

A *bull* is a triangle along with two pendant vertices that are adjacent to distinct vertices of the triangle (see Figure 3.1). A graph is *bull-free* if it does not contain a bull as an induced subgraph. Bull-free graphs generalize bipartite graphs. Chvátal and Sbihi proved that all bull-free Berge graphs are perfect [CS87].

Their proof uses two important tools: star cutsets and homogeneous pairs. A *homogeneous pair* is a pair $(A, B)$ of disjoint nonempty sets of vertices such that

1. Every vertex outside $A \cup B$ sees either all of $A$ or none of $A$, and sees either all of $B$ or none of $B$.

2. At least one of $A$ and $B$ contains at least two vertices.

3. There are at least two vertices outside $A \cup B$.

Homogeneous pairs generalize homogeneous sets, since a homogeneous set is just a homogeneous pair in which one set is empty. If $A$ and $B$ are both cliques then $(A, B)$ is a *homogeneous pair of cliques*; these generalize homogeneous cliques. Homogeneous pairs of cliques are a key idea in the structure theorems for claw-free graphs. Everett, Klein, and Reed gave an $O(mn^3)$ for finding a homogeneous pair in a general graph [EKR97]. We will give faster algorithms for finding specific types of homogeneous pairs of cliques later in this thesis.

Chvátal and Sbihi's work relies on the Star Cutset Lemma, which we already proved, and the *Homogeneous Pair Lemma*, proved by Chvátal in [CS87]:

**Lemma 3.10** (Homogeneous Pair Lemma)**.** *No minimal imperfect graph contains a homogeneous pair.*

To prove that bull-free Berge graphs are perfect, Chvátal and Sbihi proved the following structure theorem:

**Theorem 3.11.** *Let $G$ be a bull-free Berge graph. Then one of $G$ or $\overline{G}$ contains a star cutset or a homogeneous pair, or is bipartite.*

This immediately yields the result that bull-free Berge graphs are perfect, since no minimal imperfect graph is bipartite or the complement of a bipartite graph, thus no minimal imperfect graph is both bull-free and Berge. Reed and Sbihi proved a different structure theorem for bull-free Berge graphs that yields a polynomial-time detection algorithm [RS95]. As we discuss at the end of this chapter, it was not until recently that a polynomial-time detection algorithm was found for general Berge graphs.

The Homogeneous Pair Lemma is also implied by results of Olariu [Ola88] and Conforti, Cornuéjols, Gasparyan, and Vuškovič [CCGV02] (see [PS01]). Later in this thesis we will prove results that easily imply that no minimal imperfect graph contains a homogeneous pair of cliques.

## 3.5 The Strong Perfect Graph Theorem and a structural characterization of Berge graphs

The Strong Perfect Graph Conjecture was for a long time one of the most significant open problems in the field of graph theory. Chudnovsky, Robertson, Seymour, and Thomas finally announced that the conjecture is true in 2002. The proof, at a length of about 150 pages, was published in 2006 [CRST06]. Chudnovsky and Seymour [CS08c] later shortened it by about 50 pages using an even pairs approach inspired by similar work by Maffray and Trotignon [MT06]. For an idea of how the proof was found, we refer the reader to Seymour's gentle introduction to the work [Sey06]. Since $\chi = \omega$ for Berge graphs, the SPGT gives a strengthening of the Main Conjecture and the Local Strengthening for Berge graphs.

The proof of the SPGT follows from a structure theorem for Berge graphs due to Chudnovsky, Robertson, Seymour, and Thomas [CRST06]. The structure theorem was conjectured to exist by Conforti, Cornuéjols, and Vuškovič (see [Sey06]). It states, essentially, that a Berge graph $G$ must be in one of several well-understood classes of perfect graphs, or else it admits one of several structural decompositions. Thus the structure theorem is of the same flavour as the structure theorems for bull-free Berge graphs and claw-free graphs, the latter of which we discuss in the next chapter. A minimum counterexample to the SPGT cannot admit any of the decompositions in question, so the theorem follows (since it holds for the base classes of graphs).

It stands to reason, considering the Weak Perfect Graph Theorem, that the base classes of graphs should come in complementary pairs. There are three pairs:

- **Bipartite graphs** and their complements, cobipartite graphs.

- **Line graphs of bipartite graphs** and their complements.

- **Double split graphs** are defined as follows. Let $H$ be a split graph, i.e. a graph whose vertices can be partitioned into a stable set $A$ and a clique $B$. We double the vertices of $H$ such that the pairs from $A$ are adjacent and the pairs from $B$ are nonadjacent. That is, we let $V(G) = \bigcup_{v \in V(H)} \{v_1, v_2\}$. For $u \in A$ and $v \in B$, we have:

  – $u_1$ sees $u_2$ if and only $u \in A$

  – If $u, v \in A$ then $u_i$ does not see $v_j$ for $\{i, j\} \subseteq \{1, 2\}$.

  – If $u, v \in B$ then $u_i$ sees $v_j$ for $\{i, j\} \subseteq \{1, 2\}$.

  – If $u \in A$ and $v \in B$, there are two edges between $\{u_1, u_2\}$ and $\{v_1, v_2\}$. They are $u_1v_1$ and $u_2v_2$ if $u$ sees $v$ in $H$. Otherwise they are $u_1v_2$ and $u_2v_1$.

  Notice that the complement of a double split graph is again a double split graph, so to show that they are perfect we need only show that they contain no odd holes; this is easy.

If $G$ or its complement is bipartite or the line graph of a bipartite graph or a double split graph, we say that $G$ is *basic* (this definition only applies to this chapter, as we will use different types of basic graphs in the study of claw-free graphs). If a Berge graph is not basic then it yields one of two types of structural decomposition, which we define and discuss now.

### 3.5.1 2-joins

We already know that no minimal imperfect graph contains a clique cutset. One special type of clique cutset is a *1-join*. A graph $G$ admits a 1-join if we can partition the vertices of $G$ into $A_1, B_1, A_2, B_2$ such that $A_1 \cup A_2$ is a clique, $N(B_1) \subseteq A_1 \cup B_1$, and $N(B_2) \subseteq A_2 \cup B_2$ (that is, the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$ are those between $A_1$ and $A_2$). Observe that $A_1 \cup A_2$ is a clique cutset, so it is easy to see that no minimal imperfect graph (and thus no minimum counterexample to the SPGT) contains a 1-join.

Generalizing 1-joins are *2-joins*, which were introduced by Cornuéjols and Cunningham [CC85]. A 2-join is a partitioning of $V(G)$ into $V_1$ and $V_2$ with disjoint nonempty $X_i, Y_i \subseteq V_i$ for $i \in \{1, 2\}$ such that $X_1$ is complete to $X_2$ and $Y_1$ is complete to $Y_2$, and there are no other edges between $V_1$ and $V_2$.

The decomposition theorem for Berge graphs uses a special type of 2-join called a *proper 2-join*. A proper 2-join is a 2-join with the additional properties that:

- For $i \in \{1, 2\}$, every component of $G[V_i]$ intersects both $X_i$ and $Y_i$.

- For $i \in \{1, 2\}$, if $|X_i| = |Y_i| = 1$ and $G[V_i]$ is an induced path, then it has odd length $\geq 3$.

2-joins are a special case of *2-amalgams*, in which we allow a homogeneous clique adjacent to $X_1 \cup X_2 \cup Y_1 \cup Y_2$.

To show that 2-joins preserve perfection, as proved by Cornuéjols and Cunningham [CC85], we need to prove that we can find suitable $\omega(G)$-colourings of $G_1 = G[V_1]$ and $G_2 = G[V_2]$ so that we can paste them together to reach a proper $\omega(G)$-colouring of $G$. But observe that $G$ may contain a cycle of odd length even if both $G_1$ and $G_2$ are perfect, even if the 2-join is proper. So instead of insisting that $G_1$ and $G_2$ are perfect, we insist that two auxiliary graphs $G'_1$ and $G'_2$ are perfect. For $i \in \{1, 2\}$ we construct $G'_i$ from $G_i$ by adding a vertex $v_1$ with neighbourhood $X_i$, and

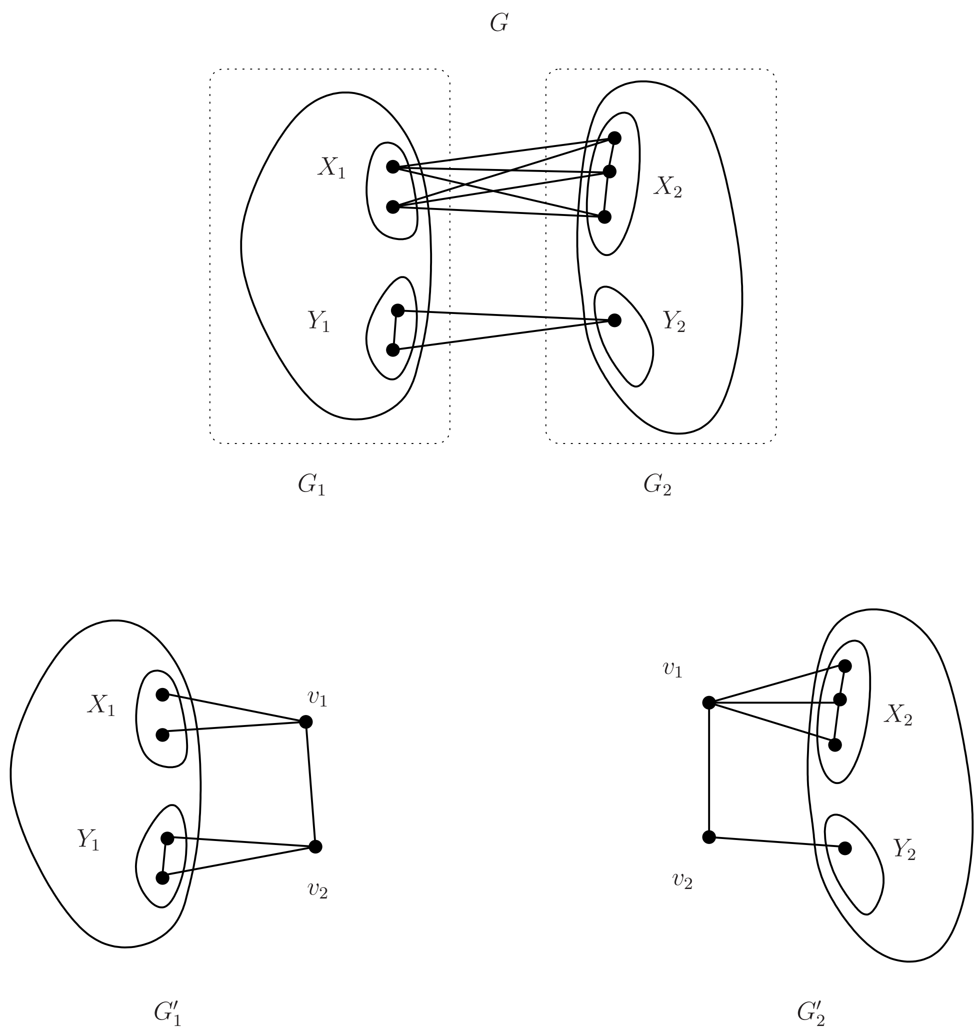


Figure 3.2: A graph $G$ admitting a 2-join into $G_1$ and $G_2$. The 2-join is built from auxiliary graphs $G_1'$ and $G_2'$. If $G_1'$ and $G_2'$ are perfect then so is $G$.

a vertex $v_2$ with neighbourhood $Y_i \cup \{v_1\}$ (see Figure 3.2). Denote the clique numbers $\omega(G_i[X_i])$ and $\omega(G_i[Y_i])$ by $p_i$ and $q_i$ respectively. Construct the graph $H_i$ from $G_i'$ by adding a vertex $u$ with neighbourhood $\{v_2\} \cup Y_i$. We can see that $H_1$ and $H_2$ are perfect, in particular because $\{v_2\} \cup Y_i$ is a star cutset in $H_i$.

The following lemma provides us with the desired colourings of $G_1$ and $G_2$.

**Lemma 3.12.** *Assume $H_i$ is perfect. For any $l \geq \omega(G_i)$ there is a proper $l$-colouring of $G_i$ in which $p_i$ colours appear on $X_i$, $q_i$ colours appear on $Y_i$, and $\max\{0, p_i + q_i - l\}$ colours appear on both $X_i$ and $Y_i$.*

*Proof.* Beginning with $H_i$, replace $v_1$ with a clique $C_1$ of size $l - p_i$, replace $v_2$ with a clique $C_2$ of size $\min\{l - q_i, p_i\}$, and replace $u$ with a clique $C_3$ of size $\max\{l - p_i - q_i, 0\}$. The resulting graph $H_i'$ is perfect by the Replication Lemma, and it is easy to confirm that $\omega(H_i') = l$.

Consider an $l$-colouring of $H_i'$ restricted to $G_i$ ($G_i$ is an induced subgraph of $H_i'$). Since $X_i$ is joined to a clique of size $l - p_i$ and $Y_i$ is joined to a clique of size $l - q_i$ (i.e. the cliques replacing $v_2$ and $u$), $p_i$ colours appear on $X_i$ and $q_i$ colours appear on $Y_i$. Since we have an $l$-colouring, at least $p_i + q_i - l$ colours appear on both $X_i$ and $Y_i$. Furthermore, the colours appearing on neither $X_i$ nor $Y_i$ are precisely the colours appearing on both $C_1$ and $C_3$, of which there must be $\max\{l - p_i - q_i, 0\}$ in a proper $l$-colouring of $H_i'$, since $H_i'[C_1 \cup X_i]$ and $H_i'[C_2 \cup C_3 \cup Y_i]$ both have clique number $l$. The result follows. □

With this lemma in hand it is now easy to prove that 2-joins preserve perfection: we argue that taking our guaranteed $\omega(G)$-colourings of $G_1$ and $G_2$, we can simply permute the colour classes and paste the colourings together to get a proper $\omega(G)$-colouring of $G$. However, different types of 2-joins are very important to us when we bound the chromatic number of quasi-line and claw-free graphs later in this thesis. With this in mind we present a proof that is less straightforward, but which better motivates our approach to 2-joins in claw-free graphs.

**Lemma 3.13.** *If $G_1'$ and $G_2'$ are perfect, then $G$ is perfect.*

*Proof.* Let $G$ be a minimum counterexample with clique number $\omega$. We claim that $G$ is minimal imperfect. For if some proper induced subgraph $G'$ of $G$ is imperfect, it admits either a 1-join (if $G'$ contains none of some $X_i$ or $Y_i$) or a 2-join resulting from perfect induced subgraphs of $G_1'$ and $G_2'$. The 1-join preserves perfection since it is a clique sum, so by minimality of $G$, $G'$ is perfect and $G$ is minimal imperfect.

It suffices, then, to prove that $G$ is $\omega$-colourable. To do this we find a stable set $S$ such that $G_1' - S$ and $G_2' - S$ are both $\omega - 1$ colourable, which proves the lemma by minimality of $G$.

We begin with an $\omega$-colouring of $G_1$ and an $\omega$-colouring of $G_2$ as provided by the previous lemma (setting $l = \omega$). Since $p_1 < \omega$ there is a colour class $S_1$ in $G_1$ that intersects $Y_1$ but not $X_1$. Suppose there is a colour class $S_2$ in $G_2$ that intersects $X_2$ but not $Y_2$. Observe that $\omega = \max\{\omega(G_1), p_1 + p_2, q_1 + q_2, \omega(G_2)\}$. Thus since $S_1 \cup S_2$ intersects both $X_1 \cup X_2$ and $Y_1 \cup Y_2$, it is easy to see that $\omega(G - (S_1 \cup S_2)) = \omega - 1$ and we are done.

So we can assume that there is no such $S_2$ in $G_2$, i.e. that every colour in $X_2$ is also in $Y_2$. By the previous lemma, $\omega = \omega(G_2) = q_2$. But this is clearly impossible since $Y_1 \cup Y_2$ is a clique and $Y_1$ is nonempty. □

We just illustrated a simple example of a very important approach, variations of which we will apply repeatedly throughout this thesis. To prove that $G$ is $\omega(G)$-colourable, we removed a stable set that lowered $\omega(G)$. In later chapters we will remove stable sets that lower $\gamma(G)$ or $\gamma_l(G)$ in order to prove that $\chi(G) \leq \gamma(G)$ or $\chi(G) \leq \gamma_l(G)$.

### 3.5.2 Balanced skew-partitions

Proper 2-joins are the first type of decomposition we need for the SPGT. The other is a specific type of *skew-partition*. A skew-partition is a partitioning of $V$ into $A$ and $B$ such that neither $G[A]$ nor $\overline{G}[B]$ is connected. Because $G[A]$ is not connected, skew-partitions are sometimes called *skew-cutsets*. If we split $A$ into $A_1$ and $A_2$ with no edges between them and we split $B$ into $B_1$ and $B_2$ with all edges between them, we say that $(A_1, A_2, B_1, B_2)$ is a *split* of a skew-partition. Given a split we define the graphs $G_1 = A_1 \cup B$ and $G_2 = A_2 \cup B$.

If a graph $G$ contains a star cutset $X$ with centre $v$, then $G[X]$ is disconnected in the complement so $G$ admits a skew-partition into $X$ and $G - X$, thus skew-partitions generalize star cutsets (and also clique cutsets). If $G$ contains a homogeneous set $H$ such that the sets $N$ and $A$ of vertices in $G - H$ which see none and all of $H$ respectively are both nonempty, then $(N \cup (H \setminus \{v\}), A \cup \{v\})$ is a skew-partition. Thus skew partitions also generalize homogeneous sets.

Chvátal [Chv85] first introduced skew-partitions upon proving the Star Cutset Lemma. He conjectured that they could not exist in a minimum counterexample to the SPGT; this conjecture remained open until the proof of the SPGT rendered it trivial [Sey06].

A heavy focus on general skew-partitions brought about difficulties in the search for a structure theorem for Berge graphs. As a response, Chudnovsky, Robertson, Seymour, and Thomas introduced *balanced skew-partitions*. A skew-partition $(A, B)$ is *balanced* if every induced path of length $\geq 2$ with endpoints in $B$ and interior in $A$ has even length, and the same property holds for $(B, A)$ in the complement of $G$. These are the second type of decomposition needed for the structure theorem on Berge graphs.

To prove that no minimum order counterexample to the SPGT contains a balanced skew-partition we must first lay some groundwork.

**Lemma 3.14** (The Colouring Lemma)**.** *Suppose $(A_1, A_2, B_1, B_2)$ is a split of a skew-partition in a minimal imperfect graph $G$. Then there do not exist $\omega(G)$-colourings $C_1$ of $G_1$ and $C_2$ of $G_2$ in which $B_1$ receives the same number of colours.*

*Proof.* For $i \in \{1, 2\}$ let $X_i$ be the union of the $k$ colour classes intersecting $B_1$ in the colouring of $G_i$ and let $X = X_1 \cup X_2$. No stable set intersects both $B_1$ and $B_2$, so $X \cap B_2 = \emptyset$. No clique intersects both $A_1$ and $A_2$, and $G - B_2$ is perfect, so $G[X]$ contains no $k+1$ clique and is therefore $k$-colourable. Our colouring of $G_i$ further shows that $G_i - X_i$ is $\omega(G) - k$ colourable. Since $G - X$ is perfect and has clique number $\max\{\omega(G_1 - X_1), \omega(G_2 - X_2)\} = \omega(G) - k$, it follows that $G$ is $\omega(G)$-colourable, contradicting the assertion that it is minimal imperfect. □

To apply the Colouring Lemma, we use a weaker version of an observation made by Hoàng [Hoà96]:

**Observation 3.15.** *Construct $G_i^*$ from $G_i$ by adding a vertex $v^*$ with neighbourhood $B$. If $G_i^*$ is perfect then there is an $\omega(G)$-colouring of $G_i$ in which precisely $\omega(B_1)$ colours appear on $B_1$.*

*Proof.* At least $\omega(B_1)$ colours appear on $B_1$ in any colouring. Denote by $\omega^*$ the size of the largest clique in $G_i^*$ containing $v^*$. Replace $v^*$ with a clique $C^*$ of size $\omega(G) - \omega^* + 1$ (this size may be zero, in which case we simply delete $v^*$). The resulting graph, which we call $G_i'$, is perfect and has clique number $\omega(G)$. Furthermore, the vertices of $B_1$ are joined to a clique of size $\omega(G) - \omega(B_1)$ contained in $B_2 \cup C^*$. It follows that in an $\omega(G)$-colouring of $G_i'$, precisely $\omega(B_1)$ colours appear on $B_1$. Deleting the vertices of $C^*$ gives us the desired colouring of $G_i$. □

The reason behind the restriction on path parity in a balanced skew-partition now makes sense, for if the skew-partition is balanced, both $G_1^*$ and $G_2^*$ will be Berge. Rather than proving that a minimal imperfect graph cannot admit a balanced skew-partition, we prove that a minimum order counterexample to the SPGT cannot admit one.

**Lemma 3.16.** *If $G$ is a minimum order imperfect Berge graph, it does not admit a balanced skew-partition.*

*Proof.* First observe that $G$ is minimal imperfect. Suppose it admits a balanced skew-partition with split $(A_1, A_2, B_1, B_2)$; we can assume that each of these sets has size at least two, otherwise either $G$ or $\overline{G}$ contains a star cutset. For $i \in \{1, 2\}$ we make $G_i^*$ from $G_i$ as in the previous observation. Both $G_1^*$ and $G_2^*$ have smaller order than $G$, and by the parity restrictions on a balanced skew-partition, both are Berge and therefore perfect.

By the previous observation, there are $\omega(G)$-colourings $C_1$ of $G_1$ and $C_2$ of $G_2$ in which precisely $\omega(B_1)$ colours appear on $B_1$. Therefore by the Colouring Lemma, $G$ is not minimal imperfect, a contradiction. □

We refer the interested reader to Reed's paper on skew-partitions [Ree08], in which he discusses the original genesis of these ideas, how some related lemmas were used in the proof of the Strong Perfect Graph Theorem, and algorithmic aspects of skew-partitions.

### 3.5.3 The decomposition theorem

In the original proof of the SPGT, one more decomposition was needed: proper homogeneous pairs. A homogeneous pair $(A, B)$ is proper if no vertex in $A$ sees all or none of $B$, no vertex in $B$ sees all or none of $A$, and there is at least one vertex in $V(G) \setminus (A \cup B)$ that sees both $A$ and $B$ (resp. $A$ but not $B$, $B$ but not $A$, neither $A$ nor $B$). The original decomposition theorem is:

**Theorem 3.17.** *For any Berge graph $G$ that is not basic, either $G$ or $\overline{G}$ admits a 2-join or a balanced skew-partition or contains a proper homogeneous pair.*

However, Chudnovsky proved that proper homogeneous pairs are unnecessary by characterizing the structure of *Berge trigraphs* [Chu06]. Homogeneous pairs and trigraphs are important concepts in the study of claw-free graphs, and we will discuss them further in Chapter 6. Chudnovsky's strengthening of the original structure theorem (both appear in [CRST06]) is quite beautiful in its simplicity:

**Theorem 3.18.** *For any Berge graph $G$ that is not basic, either $G$ or $\overline{G}$ admits a proper 2-join or a balanced skew-partition.*

In light of the results we have discussed in this section, this theorem immediately implies the Strong Perfect Graph Theorem, because no minimum order counterexample admits a 2-join or a balanced skew-partition.

This decomposition theorem is much simpler than the structure theorems for claw-free graphs for several reasons. For claw-free graphs, there are many more types of "basic" graphs, several cases of exceptions, and two separate structure theorems, depending on whether or not $\overline{G}$ is 3-colourable. Furthermore, when we use the structure theorem to prove the Main Conjecture for claw-free graphs, we rely on 2-joins in which we have a very thorough understanding of the structure of either $V_1$ or

$V_2$. We would be very interested in a general theorem proving that no minimum counterexample to the Main Conjecture admits a 2-join (or, even better, a 2-amalgam).

The structure theorem for Berge graphs suggests that it may be possible to derive an efficient combinatorial $\chi$-colouring algorithm for perfect graphs. We already showed how to optimize over perfect graphs using the ellipsoid method, but a combinatorial algorithm would be of great interest and none is yet known.

## 3.6 Recognizing Berge graphs

In spite of everything that was known about perfect graphs, there was a major question that the proof of the Strong Perfect Graph Theorem did not settle: Given a graph $G$, can we decide whether or not $G$ is Berge in polynomial time?

We have already shown that this is possible for many classes of perfect graphs, for example chordal graphs, in which we forbid all holes. The situation is much more complicated when we only forbid holes of even length. Conforti, Cornuéjols, Kapoor, and Vušković proved a structural decomposition theorem for even-hole-free graphs [CCKV02a] and gave polynomial-time algorithms to decide whether or not a graph contains an even hole, and find an even hole if one exists [CCKV02b]. Chudnovsky, Kawarabayashi, and Seymour improved on this result, giving a simpler and more general algorithm that does not use the structure theorem [CKS04]. In an analogue of Fulkerson and Gross' characterization of chordal graphs in terms of simplicial vertices, Addario-Berry, Chudnovsky, Havet, Reed, and Seymour proved that every even-hole-free graph contains two nonadjacent bisimplicial vertices (and therefore has a *bisimplicial ordering*) [ACH+08].

It is still unknown whether or not the problem of deciding whether or not a graph $G$ is odd-hole-free is *NP*-complete. However, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković found a polynomial-time algorithm for recognizing Berge graphs [CCL+05]. Surprisingly, their algorithm does not use the structure theorem for Berge graphs. Given a graph $G$, the algorithm either decides that $G$ is not Berge, or that $G$ contains no odd holes. This does not tell us whether or not $G$ is odd-hole-free, since the algorithm can simply return "not Berge" for a graph regardless of whether or not it contains an odd hole. However, running the algorithm on both $G$ and $\overline{G}$ will tell us whether or not $G$ is Berge.

This algorithm does not seek out proper 2-joins and balanced skew-partitions as per the structure theorem for Berge graphs. In fact, it was not until later that Trotignon [Tro08] proved that balanced skew-partitions can be detected in a Berge graph in polynomial time. This result is based on Cornuéjols and Cunningham's result that proper 2-joins can be found in polynomial time [CC85], and uses the structure theorem for Berge graphs. Trotignon's detection algorithm does not find a specific balanced skew-partition if there is one. In the same paper he proved that the problem of detecting balanced skew-partitions is *NP*-complete for general graphs. In contrast, de Figueiredo, Klein, Kohayakawa, and Reed gave a polynomial-time ($O(n^{101})$) algorithm for finding a skew-partition in a graph $G$ [dFKKR00]. A much faster ($O(n^4m)$) algorithm was later given by Kennedy and Reed [KR08a].

# Chapter 4

# Proving the Conjecture When $\chi$ and $\chi_f$ Are Close

As discussed in Chapter 2, $\lceil \chi_f(G) \rceil \leq \gamma_l(G)$ so if $\chi(G) \leq \lceil \chi_f(G) \rceil$ then the Local Strengthening holds. In this chapter, we use this fact to prove the Main Conjecture for two classes of graphs: circular interval graphs (defined below) and line graphs. Our focus on these graphs is motivated by the fact that these classes are used in the decomposition theorem for claw-free graphs. They are used both as the base classes into which we decompose and in defining some of the operations used to decompose.

Niessen and Kind proved the *round-up property*, that $\chi \leq \lceil \chi_f \rceil$, for circular interval graphs [NK00], and hence proved that the Local Strengthening holds for these graphs. In Section 4.1, we present a corollary of their result which we will use in bounding the chromatic number of quasi-line graphs.

Seymour and Goldberg conjectured that the round-up property holds for a line graph $G$ of $H$ provided the fractional chromatic number of $G$ exceeds $\Delta(H)$. Since $\omega(G) \geq \Delta(H)$, this easily implies the Local Strengthening for line graphs (as a connected graph which satisfies $\gamma_l(G) < \omega(G) + 1$ is a clique and so satisfies the Local Strengthening). However, this famous conjecture has not yet been proved and the Local Strengthening remains open for line graphs.

It is known that the round-up property holds for the line graph $G$ of $H$ provided $\chi(G) > 1.1\Delta(G) + 0.8$. In Section 4.2, we use this partial result towards the Seymour-Goldberg Conjecture to prove the Main Conjecture for line graphs.

## 4.1 Linear and circular interval graphs

Two well-known classes of claw-free graphs are linear interval graphs and circular interval graphs. Circular interval graphs are sometimes called *proper circular arc graphs* and are not to be confused with circular arc graphs, which are not necessarily claw-free. Linear interval graphs are often known as *unit interval graphs*. As we shall see, these two classes of graphs play an important role in the structure theorems for claw-free graphs.

A *linear interval representation* of a graph $G = (V, E)$ consists of a point on the real number line corresponding to each vertex, along with a set of closed intervals such that two vertices $u$ and $v$ of the graph are adjacent precisely if there is an interval containing the two points corresponding to $u$ and $v$. Obviously in a representation with the fewest number of intervals, none contains another.

A *linear interval graph* is a graph for which there is a linear interval representation. These graphs can be both recognized and represented in linear time [DHH96]. Linear interval graphs are chordal and therefore perfect, and they can be $\omega$-coloured in linear time by colouring the vertices greedily, moving from left to right along the real line.

A *circular interval representation* of a graph $G = (V, E)$ consists of a point on the boundary of the unit circle corresponding to each vertex, along with a set of closed intervals on the boundary of the unit circle such that two vertices of $G$ are adjacent precisely if there is an interval containing both points associated with the vertices (again we assume no interval contains another). A *circular interval graph* is a graph for which there is a circular interval representation. Like linear interval graphs, circular interval graphs can be recognized and represented in linear time [DHH96].

Henceforth, when we are given a circular (resp. linear) interval graph on $n$ vertices we will assume the vertices are labelled $v_1, \dots, v_n$ in clockwise (resp. left-to-right, i.e. ascending) order. Given a linear interval representation of a linear interval graph $G$ in which the points corresponding to $v_1, \dots, v_n$ appear in left-to-right order, we can easily find a circular interval representation of $G$ in which the points corresponding to $v_1, \dots, v_n$ appear in clockwise order, so there is no ambiguity.

As mentioned earlier, Niessen and Kind proved that $\chi = \lceil \chi_f \rceil$ for circular interval graphs, implying the Local Strengthening:

**Theorem 4.1.** *For any circular interval graph $G$, $\chi(G) \leq \gamma_l(G)$.*

Teng and Tucker [TT85] gave an algorithm for optimally colouring a circular interval graph in polynomial time; Shih and Hsu [SH89] later improved the complexity analysis:

**Theorem 4.2.** *Given a circular interval graph $G$, we can $\chi(G)$-colour $G$ in $O(n^{3/2})$ time.*

We now use Niessen and Kind's result to construct an integer colouring of a linear interval graph that emulates a given fractional colouring at the far left and right ends of the graph.

### 4.1.1 Fractional and integer colourings in linear interval graphs

When we say that a linear interval graph $G$ has *end-cliques* $L$ and $R$, we mean that $G$ has specified cliques $L$ and $R$ consisting of the $|L|$ leftmost and $|R|$ rightmost vertices in the linear interval representation, respectively. We use the round-up property of circular interval graphs to prove that we can closely emulate any fractional colouring of a linear interval graph with an integer colouring, with respect to $L$ and $R$. This lemma will be of use to us in relating $\chi_f$ and $\chi$ for quasi-line graphs.

**Lemma 4.3.** *Let $G$ be a linear interval graph with end-cliques $L$ and $R$, and let $c$ and $w$ be nonnegative integers. If there is a fractional $c$-colouring of $G$ such that the weight of the stable sets intersecting both $L$ and $R$ is precisely $w$, then there is an integer $c$-colouring of $G$ such that exactly $w$ colours intersect both $L$ and $R$.*

*Proof.* Consider such a fractional $c$-colouring. Let $D$ be the set of colours appearing in both $L$ and $R$, let $X$ be the set appearing in $L$ but not $R$, $Y$ the set appearing in $R$ but not $L$, and $Z$ the set appearing in neither $L$ nor $R$.

Observe that since $c$, $w$, $|L|$ and $|R|$ are integers, all of $wt(D) = w$, $wt(X) = |L| - w$, $wt(Y) = |R| - w$ and $wt(Z) = c - |L| - |R| + w$ are integers. We construct a circular interval graph $G'$ based on the fractional colouring of $G$.

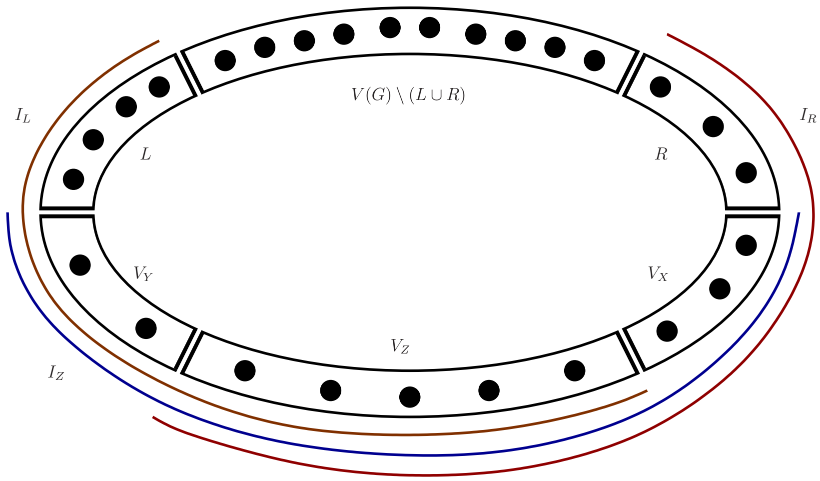


Figure 4.1: $G'$, with the vertices of $G$ along the top.

Say the $n$ vertices of $G$ are $v_1, \ldots, v_n$, left-to-right – thus in a circular interval representation of $G$ they appear in clockwise order. To construct $G'$ from $G$, we add cliques $V_X$, $V_Z$, and $V_Y$ of size $wt(X)$, $wt(Z)$, and $wt(Y)$ respectively, in clockwise order starting after $v_n$. To determine their adjacency to each other and the vertices of $G$, we make three new intervals $I_R$, $I_Z$, and $I_L$. These span, respectively: $\{v_i \mid n - |R| < i\} \cup V_X \cup V_Z$, $V_X \cup V_Z \cup V_Y$, and $\{v_i \mid i \leq |L|\} \cup V_Y \cup V_Z$ (see Figure 4.1). Thus $V_X$ is complete to $R$ and $V_Z$, $V_Z$ is complete to $V_X$ and $V_Y$, and $V_Y$ is complete to $V_Z$ and $L$. Since $I_L$ and $I_R$ define $c$-cliques and $I_Z$ defines a $c - w$ clique, it is easy to see that $G'$ is a circular interval graph with clique number $c$, hence both the fractional chromatic number and the chromatic number of $G'$ are at least $c$.

Now we give a fractional $c$-colouring of $G'$. On the vertices belonging to $G$, we keep our initial fractional colouring. We can then cover $V_X$ with the colours in $X$, $V_Z$ with the colours in $Z$, and $V_Y$ with the colours in $Y$. Hence $\chi_f(G') = c$ and so by Niessen and Kind's result that for circular interval graphs, $\chi = \lceil \chi_f \rceil$, we know that $\chi(G') = c$. So consider an integer $c$-colouring of $G'$. We claim that on the vertices of $G$ this gives us an integer $c$-colouring of $G$ containing exactly $w$ colours that appear in both $L$ and $R$.

Suppose fewer than $w$ colours appear in both $L$ and $R$. Then there are more than $|L| + |R| - w$ colours that cannot appear in $V_Z$. But $V_Z$ is a clique of size $c - (|L| + |R| - w)$, contradicting the fact that we have a proper $c$-colouring. Now suppose more than $w$ colours appear in both $L$ and $R$. Then none of these colours can appear in $V_X \cup V_Z \cup V_Y$. But $V_X \cup V_Z \cup V_Y$ is a clique of size $c - w$ so again we cannot have a proper $c$-colouring. Thus exactly $w$ colours appear in both $L$ and $R$. Restricting this colouring of $G'$ to the vertices of $G$ gives us the desired colouring. □

## 4.2 Line graphs

Line graph colouring is of particular interest because a colouring of a line graph corresponds to an edge colouring of the underlying multigraph. Much of the interest in quasi-line graphs and claw-free graphs comes from the fact that they are both natural generalizations of line graphs. This link to edge colouring also makes line graphs easy to colour using techniques from that domain. So it is not surprising that line graphs were the first substantial subclass of claw-free graphs for which the Main Conjecture was proved. This was done by the author with Reed and Vetta [KRV07]. Before we prove this result we need to review some past results on colourings of line graphs, i.e. edge colourings of multigraphs.

### 4.2.1 Fractional and integer colourings in line graphs

The *chromatic index* of a graph or multigraph $H$, written $\chi'(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi'_f(H)$ is equal to the fractional chromatic number of $L(H)$. Holyer proved that determining the chromatic index of an arbitrary graph is *NP*-complete [Hol81], so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs (and, for that matter, claw-free graphs).

Vizing's Theorem for multigraphs [Viz64] bounds the chromatic index of a multigraph in terms of its maximum degree and multiplicity $d$, stating that $\Delta(H) \leq \chi'(H) \leq \Delta(H) + d$, where $d$ is the maximum number of edges between any two vertices in $H$. Both bounds are achievable, but a more meaningful bound should consider other invariants of $H$.

Of course, $\chi'(H)$ is always bounded below by $\chi'_f(H)$. Let $w$ be a non-negative weighting on the edges of $H$. Duality tells us that given a non-negative weighting $w$ on the edges of $H$ such that for every matching $M$ in $H$, $\sum_{e \in M} w(e) \leq 1$, $\chi'_f(H) \geq \sum_{e \in E(H)} w(e)$. Two such weightings give us lower bounds. In the first, we assign a weight of 1 to each edge incident to a given vertex $v$ of maximum degree; every other edge is assigned weight 0. In the second, we take an induced subgraph $W$ of $H$ and assign to each edge of $W$ a weight of $1/\lfloor |V(W)|/2 \rfloor$; other edges of $H$ are assigned weight 0. Edmond's theorem for matching polytopes (presented in [Edm65a], also mentioned in [Kah00]) tells us that the greater of these lower bounds is tight, so setting

$$\Gamma(H) = \max \left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

we have

$$\chi'_f(H) = \max\{\Delta(H), \Gamma(H)\}. \tag{4.1}$$

The long-standing conjecture posed by Seymour [Sey79] and Goldberg [Gol73] proposes that this gives us a very nearly tight bound on the chromatic index of a multigraph:

**Seymour-Goldberg Conjecture.** *For a multigraph $H$ for which $\chi'(H) > \Delta(H) + 1$, $\chi'(H) = \lceil \Gamma(H) \rceil$.*

Kahn [Kah96] proved that this bound holds asymptotically, i.e. that $\chi'(H) \leq (1 + o(1))\chi'_f(H)$. He later proved that in fact, the fractional chromatic index asymptotically agrees with the *list chromatic index* [Kah00]. More useful to us in proving the Main Conjecture is the following algorithmic approximation result:

**Theorem 4.4** (Nishizeki and Kashiwagi [NK90])**.** *For any multigraph $H$, $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H) + 0.8 \rfloor, \lceil \Gamma(H) \rceil\}$. Furthermore in $O(|E(H)| \cdot |V(H)|)$ time we can find an edge colouring of $H$ that achieves this bound.*

Caprara and Rizzi [CR98] later improved the term $\lfloor 1.1\Delta(H) + 0.8 \rfloor$ in the bound to $\lfloor 1.1\Delta(H) + 0.7 \rfloor$, and Scheide [Sch07a] recently improved it to $\lfloor \frac{15}{14}\Delta(H) + \frac{12}{14} \rfloor$. Note that this implies the

Seymour-Goldberg Conjecture for any multigraph $H$ with $\Delta(H) \leq 15$, since in this case we have $\lfloor \frac{15}{14}\Delta(H) + \frac{12}{14} \rfloor \leq \Delta(H) + 1$.

With these results in hand we can prove the Main Conjecture for line graphs.

### 4.2.2 Proving the Main Conjecture for line graphs

In this section we prove the main result of [KRV07]:

**Theorem 4.5.** *For any line graph $G = L(H)$, $\chi(G) \leq \gamma(G)$.*

To prove this we use the standard approach described in Section 2.6: Assuming $G$ is a minimum counterexample, we find and remove a stable set $S$ such that $\gamma(G - S) < \gamma(G)$. If we cannot find such an $S$, we somehow exploit this fact to prove $\chi(G) \leq \gamma(G)$ directly.

For line graphs, we can find the desired $S$ unless $\Delta(H)$ and $\Delta(G)$ are far apart, in which case the following easy corollary of Theorem 4.4 allows us to prove directly that the Main Conjecture holds.

**Lemma 4.6.** *If $G$ is the line graph of a multigraph $H$ and $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, then $\chi(G) \leq \gamma(G)$.*

*Proof.* We know that $\chi_f(G) = \max\{\Delta(H), \Gamma(H)\}$, so Theorem 4.4 tells us that $\chi(G) \leq \max\{\lfloor 1.1\Delta(H) + 0.8 \rfloor, \lceil \chi_f(G) \rceil\}$. We know that the Main Conjecture holds fractionally for $G$ by Theorem 2.10, so $\lceil \chi_f(G) \rceil \leq \gamma(G)$. Since $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, we have $\gamma(G) = \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil \geq \lceil \frac{1}{2}(\frac{3}{2}\Delta(H) + \Delta(H)) \rceil \geq \lceil \frac{5}{4}\Delta(H) \rceil \geq \lfloor 1.1\Delta(H) + 0.8 \rfloor$, so $\gamma(G) \geq \chi(G)$ and we are done. □

We can now proceed to the case in which $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. In this case we will find a stable set $S$ hitting every maximum clique in $G$. This stable set corresponds to a matching $M$ in $H$. First we introduce some notation. For vertices $u$ and $v$ in a multigraph, the *multiplicity* of $uv$, denoted $\mu(u, v)$, is the number of edges between $u$ and $v$. We denote by $tri(H)$ the maximum number of edges in a triangle in $H$.

*Proof of Theorem 4.5.* Let $G$ be a minimum counterexample to the theorem; we know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Note the following facts that relate the invariants of $G$ and $H$:

**Fact 1.** $\Delta(G) = \max_{v_1v_2 \in E_H}\{\deg(v_1) + \deg(v_2) - \mu(v_1, v_2) - 1\}$.

**Fact 2.** $\omega(G) = \max\{\Delta(H), tri(H)\}$.

We will find a maximal matching $M$ whose removal from $H$ lowers both $\Delta(H)$ and $\max\{\Delta(H), tri(H)\}$. To this end, let $S_\Delta$ be the set of vertices of degree $\Delta(H)$ in $H$ and let $T$ be the set of triangles in $H$ that contain $\max\{\Delta(H), tri(H)\}$ edges. It is instructive to consider how the elements of $T$ interact.

**Observation 4.7.** *If two triangles of $T$ intersect in exactly the vertices $a$ and $b$ then $ab$ has multiplicity greater than $\Delta(H)/2$.*

*Proof.* For any edge $e$ of $H$ between $a$ and $b$, the degree of the corresponding vertex of $G$ is at least $2\Delta(H) - \mu(a, b)$. □

**Observation 4.8.** *If $abc$ is a triangle of $T$ intersecting another triangle $ade$ of $T$ in exactly the vertex $a$ then $\mu(b, c)$ is greater than $\Delta(H)/2$.*

*Proof.* The degree of a vertex of $G$ corresponding to an edge between $a$ and $d$ is at least $2\Delta(H) - \mu(b,c) - 1$. $\square$

**Observation 4.9.** *If there is an edge of $H$ joining two vertices $a$ and $b$ of $S_\Delta$ then $\mu(a,b) > \Delta(H)/2$.*

Guided by these observations, we let $T'$ be those triangles in $T$ that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and $S'_\Delta$ be those elements of $S_\Delta$ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding $S'_\Delta$ and $T'$ will serve us well. Recall that for a set $S$ of vertices we denote the union of the vertices' neighbourhoods by $N(S)$.

**Observation 4.10.** *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

*Proof.* It follows from Observation 4.9 that $S'_\Delta$ is a stable set. This means that $S$ and $N(S)$ are disjoint, and given $S \subseteq S'_\Delta$ there are $|S|\Delta(H)$ edges between $S$ and $N(S)$; the result follows from the fact that no vertex in $N(S)$ has degree $\geq \Delta(H)$. $\square$

**Observation 4.11.** *If an edge $ab$ in $H$ has exactly one endpoint in a triangle $bcd$ of $T'$, then the degree of $a$ is less than $\Delta(H)$.*

*Proof.* Any vertex in $G$ corresponding to an edge between $a$ and $b$ has degree at least $\deg(a) - 1 + \Delta(H) - \mu(c,d)$, and $\mu(c,d) \leq \Delta(H)/2$. $\square$

**Observation 4.12.** *If an edge $ab$ in $H$ has exactly one endpoint in a triangle $bcd$ of $T'$, then $\mu(a,b) \leq \Delta(H)/2$.*

*Proof.* The degree of any vertex in $G$ corresponding to an edge between $b$ and $c$ has degree at least $\mu(a,b) + \Delta(H) - 1$. $\square$

**Observation 4.13.** *For any vertex $v$ with two neighbours $u$ and $w$, $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

*Proof.* An edge between $u$ and $v$ is incident to at least $\deg(u) + \mu(vw) - 1$ other edges. $\square$

Finally, we state Hall's Theorem (see [Hal35]), a fundamental result on matchings in bipartite graphs.

**Hall's Theorem.** *Let $G$ be a bipartite graph with vertex set $V = (A, B)$. There is a matching that hits every vertex in $A$ precisely if for every $S \subseteq A$, $|N(S)| \geq |S|$.*

It is now straightforward to show that the desired matching exists. We begin with a matching $M$ consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4.10 tells us that we can apply Hall's Theorem to get a matching in $H$ that hits $S'_\Delta$; Observation 4.13 dictates that this matching cannot hit $M$, so the union $M'$ of these two matchings is a matching in $H$ that hits $S_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in $H$ or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this $M'$ to contain an edge of every triangle in $T'$? Observations 4.7 and 4.8 tell us that any two triangles in $T'$ are vertex-disjoint, so our only worry is that $M'$ already hits two vertices of some triangle in $T'$. Observations 4.9, 4.11 and 4.12 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree

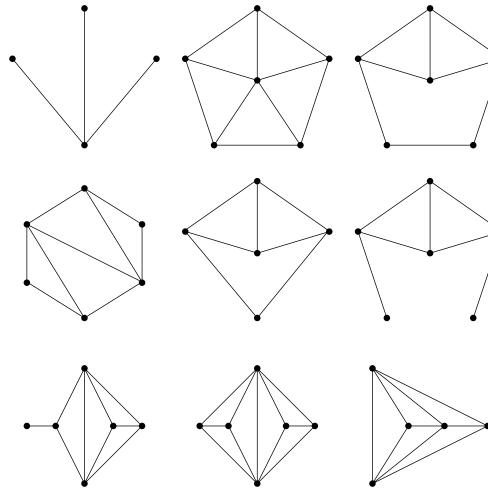

Figure 4.2: The 9 forbidden subgraphs for line graphs of simple graphs.

$\Delta(H)$. We can therefore extend $M'$ to contain an edge of every triangle in $T'$. The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. □

### 4.2.3 Algorithmic considerations

We can determine whether or not a graph $G$ is quasi-line or claw-free by looking at the structure of each neighbourhood in $G$, but for line graphs this is not the case. Beineke [Bei70] characterized line graphs of simple graphs in terms of nine forbidden subgraphs, one of which is the claw (see Figure 4.2). This characterization gives us a trivial $O(n^6)$ algorithm for testing whether or not $G$ is the line graph of a simple graph. Lehot [Leh74] and Roussopoulos [Rou73] independently gave $O(m)$-time algorithms that find a graph $H$ for which $G = L(H)$ if it exists, and determine that $G$ is not a line graph of a simple graph otherwise.

We can assume that if two vertices $u$ and $v$ are together in a homogeneous clique in a line graph $G$, then they correspond to parallel edges (i.e. edges between the same two endpoints) in the base multigraph $H$. Thus to find $H$ quickly, we first find the maximal homogeneous cliques in $G$ – they are disjoint. Contracting each homogeneous clique down to a single vertex gives us a graph $G'$ which is a line graph of the simple graph $H'$ underlying $H$. We can find $H'$ from $G'$ in $O(m)$ time, then construct $H$ from $H'$ using our knowledge of the homogeneous cliques. Thus we can construct $H$ from $G$ in $O(m)$ time if we can find the maximal homogeneous cliques of $G$ in $O(m)$ time. And we can do this using the *modular decomposition tree* of the graph, which can be computed in linear time [CH94, MS94, MdP04].

With $H$ in hand, a polytime algorithm for $\gamma(G)$-colouring $G$ follows naturally from our proof of Theorem 4.5. Theorem 4.4 implies that using Nishizeki and Kashiwagi's result, we can $\gamma(G)$ colour a line graph $G = L(H)$ in $O(|E(H)| \cdot |V(H)|)$ time if $\Delta(G) \leq \frac{3}{2}\Delta(H) - 1$. If $\Delta(G) > \frac{3}{2}\Delta(H) - 1$ we find a maximal matching $M$ in $H$ whose removal lowers $\omega(G)$ and therefore lowers $\gamma(G)$. To do this we first hit high-multiplicity (i.e. $> \Delta(H)/2$) edges, then we find a matching saturating $S'_\Delta$. Finally we hit the outstanding triangles of weight at least $\max\{\Delta(H), tri(H)\}$. The difficult part of this process is saturating $S'_\Delta$, but we can reduce this to the problem of finding a maximum matching in a bipartite graph between $S'_\Delta$ and its neighbourhood. Using Hopcroft and Karp's

$O(\sqrt{n}m)$ algorithm for finding a maximum matching in a bipartite graph [HK73], we can find $M$ in $O(|V(H)|^{1/2}|E(H)|) = O(|E(H)|^{3/2})$ time.

Thus we can find a $\gamma(G)$ edge colouring of $H$ in $O(|V(G)|^{5/2})$ time as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching $M$ from $H$ as in the proof of Theorem 4.5 (and let it be a colour class).

2. Employ Nishizeki and Kashiwagi's algorithm to complete the edge colouring of $H$.

This gives us the following algorithmic version of Theorem 4.5:

**Theorem 4.14.** *Let $G$ be a line graph on $n$ vertices. Then in $O(n^{5/2})$ time we can find a proper $\gamma(G)$-colouring of $G$.*

**Remark:** In [KR08b] the complexity of our colouring algorithm was stated as $O(n^{7/2})$. This was obtained by treating the running time of Hopcroft and Karp's matching algorithm as $O(|V(H)^{5/2}|) = O(|V(G)|^{5/2})$, when in fact the running time of the algorithm is $O(|V(H)|^{1/2}|E(H)|) = O(|V(G)|^{3/2})$.

In the next chapter we will reduce the problem of $\gamma$-colouring quasi-line graphs to the problem of $\gamma$-colouring line graphs.

# Part III

# Quasi-line Graphs and Thickenings

If every vertex in a graph $G$ is bisimplicial, i.e. its neighbours can be covered by two cliques, then the graph is clearly claw-free. Such a graph is *quasi-line*. The next three chapters concern these graphs. In the next chapter we describe their structure. In the process we introduce *compositions of strips*, a method of composition which also plays a key role in the description of the structure of claw-free graphs. We also describe two ways to expand vertices and edges in quasi-line graphs: *augmentations*, which are used for Berge quasi-line graphs, and *thickenings*, which are more general and are used for all claw-free graphs.

Thickenings result in homogeneous pairs of cliques. In Chapter 6 we explain how we can manipulate homogeneous pairs of cliques to our advantage. We introduce a reduction that we can apply to certain types of them. This reduction preserves the chromatic number and fractional chromatic number and leaves a graph whose structure is easier to characterize. These irreducible or *skeletal* graphs contain only very simple homogeneous pairs of cliques, and we can restrict our attention to them when proving our colouring results in Chapters 7 and 10.

In Chapter 7 we use a refined structure theorem for skeletal quasi-line graphs to bound the chromatic number of all quasi-line graphs. We extend two bounds from line graphs to quasi-line graphs. First we prove the Main Conjecture for quasi-line graphs, then we prove that $\chi_f$ and $\chi$ agree asymptotically for these graphs:

**Theorem III.1.** *For any quasi-line graph $G$, $\chi(G) \leq \gamma(G)$.*

**Theorem III.2.** *For any quasi-line graph $G$, $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$.*

# Chapter 5

# The Structure of Quasi-line Graphs

In this chapter we describe the structure of quasi-line graphs, i.e. graphs in which every vertex is bisimplicial. We begin by describing the structure of Berge quasi-line graphs. They arise from two base classes that foreshadow those used in the construction of general quasi-line graphs. Quasi-line graphs are built from circular interval graphs and line graphs via a composition operation that will also be useful for more general classes of claw-free graphs. To expand vertices and edges in quasi-line graphs and claw-free graphs we use *thickenings* and their more specific precursor, *augmentations*.

After introducing the necessary machinery we present a version of Chudnovsky and Seymour's structure theorem for quasi-line graphs that we will sharpen in the next chapter.

## 5.1 Berge quasi-line graphs

If a claw-free graph $G$ contains a vertex $v$ that is not bisimplicial, then $G[N(v)]$ contains an odd antihole of length at least five, so $G$ is not Berge. Thus the Berge claw-free graphs are precisely the Berge quasi-line graphs. The first step towards characterizing Berge quasi-line graphs was a theorem of Chvátal and Sbihi [CS88], which we state now.

**Definition 5.1.** *A graph $G$ is* elementary *if its edges can be 2-coloured such that for distinct $t, u, v \in V$ with $t$ nonadjacent to $v$ but $u$ adjacent to both $t$ and $v$, the edges $tu$ and $uv$ receive different colours.*

**Definition 5.2.** *A graph $G$ is* peculiar *if it can be obtained by the following construction. Begin with three non-complete pairwise disjoint cobipartite graphs $(A_1, B_2)$, $(A_2, B_3)$, and $(A_3, B_1)$, and add all edges between them*[1]*. Now add three pairwise disjoint nonempty cliques $K_1$, $K_2$, $K_3$, and add all possible edges between $K_i$ and $A_j \cup B_j$ for $i \neq j$ (see Figure 5.1).*

**Theorem 5.3** ([CS88])**.** *Any Berge quasi-line graph contains a clique cutset or is elementary or peculiar.*

This result was refined by Maffray and Reed, who characterized the structure of elementary graphs [MR99]. To do this they introduced *augmentations*, which we describe now.

Let $M$ be a nonempty matching in $G$, no edge of which is in a triangle. We construct $G'$ as follows. The vertices of $G'$ are partitioned into $|V(G)|$ disjoint nonempty cliques $\{I(v) \mid v \in V(G)\}$.

[1]Since none of these cobipartite graphs is a clique, all the cliques $A_i$ and $B_i$ are nonempty.

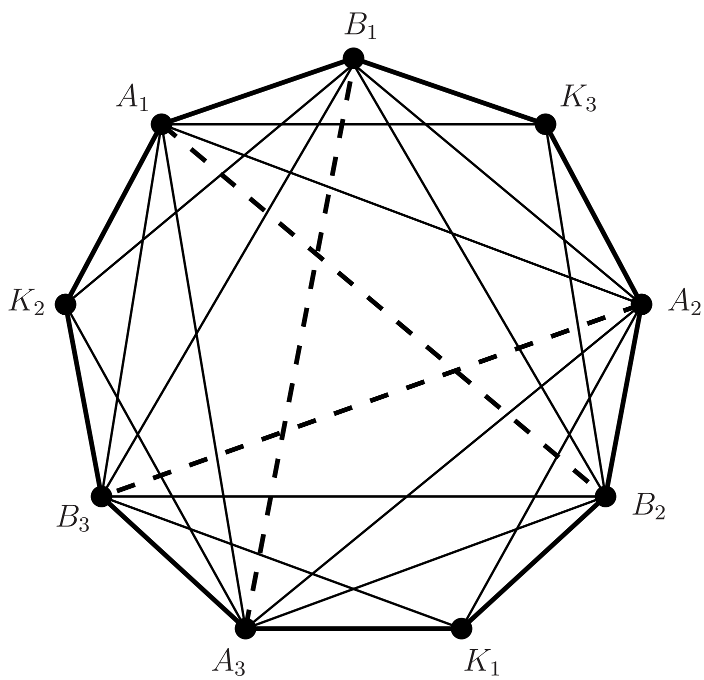


Figure 5.1: A peculiar graph. Circles represent cliques. Two cliques with a solid (resp. dashed) line between them are complete (resp. not complete). Cliques with no line between them are anticomplete.

If $v \in V(G)$ is not the endpoint of an edge in $M$, then $|I(v)| = 1$. If $u$ and $v$ are nonadjacent then $I(u)$ is anticomplete to $I(v)$. If $uv \in M$ then $I(u)$ is neither complete nor anticomplete to $I(v)$. If $uv \in E(G) \setminus M$ then $I(u)$ is complete to $I(v)$. We say that $G'$ is an *augmentation* of $G$ under $M$. See Figure 5.2

Maffray and Reed [MR99] proved:

**Theorem 5.4.** *A graph $G$ is elementary precisely if it is an augmentation of a line graph of a bipartite multigraph.*

Combining this with Theorem 5.3 we obtain:

**Theorem 5.5.** *Any Berge quasi-line graph contains a clique cutset or is peculiar or is an augmentation of a line graph of a bipartite multigraph.*

Augmentations have many useful properties. Suppose $G'$ is an augmentation of $G$ under $M$. Then:

- $G$ is a proper induced subgraph of $G'$.
- $G'$ is quasi-line precisely if $G$ is quasi-line. $G'$ is claw-free precisely if $G$ is claw-free.

  This follows easily from the fact that no edge of $M$ is in a triangle.
- For $uv \in M$, $(I(u), I(v))$ is a homogeneous pair of cliques unless both $I(u)$ and $I(v)$ are singletons.
- Suppose $G$ is claw-free. Then for $uv \in M$, $G$ admits a 2-join $((N_G(u)-v, N_G(v)-u), (\{u\}, \{v\}))$ and $G'$ admits a 2-join $((N_G(u) - v, N_G(v) - u), (I(u), I(v)))$.

  To see this, note that $u$ and $v$ have no common neighbours, thus $N_G(u) - v$ and $N_G(v) - u$ are disjoint cliques (see Figure 5.2).

With these facts in hand we can easily prove the Strong Perfect Graph Conjecture for quasi-line graphs (and therefore claw-free graphs). Of course, this immediately implies both the Main Conjecture and the Local Strengthening for Berge quasi-line graphs.

**Theorem 5.6.** *Every Berge quasi-line graph is perfect.*

*Proof.* Observe that the class of Berge quasi-line graphs is hereditary. Thus if the theorem is not true there must be a minimal imperfect Berge quasi-line graph $G$. We know that $G$ cannot contain a clique cutset, so it is elementary or peculiar.

Suppose $G$ is peculiar. Then $A_1 \cup B_1 \cup A_2 \cup B_2$ is a star cutset with a centre in $B_1$; it separates $K_3$ from the rest of the graph. This contradicts the Star Cutset Lemma, so $G$ cannot be peculiar.

Therefore $G$ is elementary. It cannot be a line graph of a bipartite multigraph, because these graphs are perfect. Thus $G$ arises via augmentation and therefore contains a homogeneous pair of cliques. Thus contradicts the Homogeneous Pair Lemma. Therefore $G$ cannot exist. □

Our use of homogeneous pairs of cliques in this proof foreshadows their importance throughout the rest of this thesis. For many classes of claw-free graphs, we can simplify the structure by assuming that no homogeneous pair of cliques exists. For Berge quasi-line graphs, we get the following:

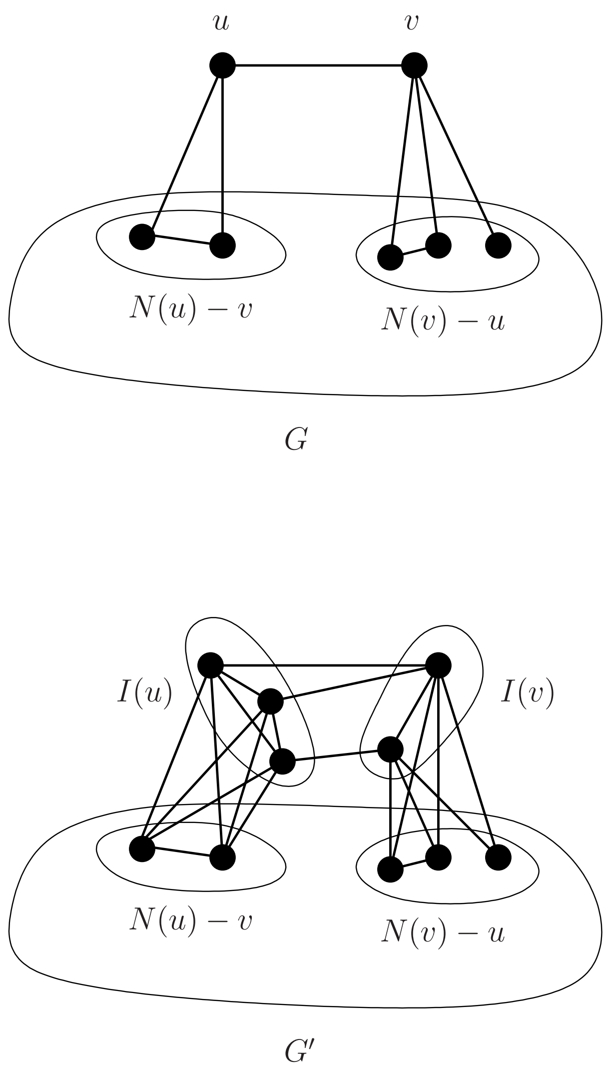


Figure 5.2: The graph $G'$ is an augmentation of $G$ under $M = \{uv\}$. If $G$ is claw-free then both $N(u) - v$ and $N(v) - u$ are cliques and a 2-join arises.

**Theorem 5.7.** *Let $G$ be a Berge quasi-line graph containing no homogeneous pair of cliques. Then $G$ is either a circular interval graph on nine vertices, or the line graph of a bipartite graph.*

*Proof.* Suppose that $G$ contains no homogeneous pair of cliques. Observe that since $G$ has at least four vertices, then it cannot contain a homogeneous clique, otherwise it would contain a homogeneous pair of cliques, one of which is a singleton. If $G$ is peculiar, it follows that $K_1$, $K_2$, and $K_3$ are singletons. And since none of $(A_1, B_2)$, $(A_2, B_3)$, or $(A_3, B_1)$ is a homogeneous pair of cliques, the sets $A_i$ and $B_i$, $1 \leq i \leq 3$ are also singletons. Thus $G$ must be a circular interval graph on nine vertices (see Figure 5.1) in which dashed lines represent either adjacent or nonadjacent vertices.

If $G$ is an augmentation of a line graph of a bipartite multigraph, then since it contains no homogeneous pair of cliques, $I(v)$ must be a singleton for every vertex $v$ of the base line graph. It follows that $G$ is a line graph of a bipartite multigraph. □

So the assumption that a graph contains no homogeneous pair of cliques simplifies the description of Berge quasi-line graphs. The same is true for describing quasi-line graphs and claw-free graphs. We would like to take advantage of this fact by somehow reducing the problem of colouring claw-free graphs to the problem of colouring claw-free graphs containing no homogeneous pair of cliques. We cannot do this, but in the next chapter we will look closely at reducing homogeneous pairs of cliques, and find that we can simplify homogeneous pairs nicely without sacrificing too much information about colourings.

**Remark:** Theorem 5.6 was first proved in 1976 by Parthasarathy and Ravindra [PR76]. An alternative proof was given by Giles, Trotter, and Tucker in 1984 [GJT84]. Reed presents a simplified version of Parthasarathy and Ravindra's proof in his Ph.D. thesis [Ree86]. These proofs, which came before Chvátal and Sbihi's decomposition theorem, use the Weak Perfect Graph Theorem.

## 5.2 Compositions of strips

To describe the structure of quasi-line graphs we first need to generalize augmentations of line graphs. We do this using *compositions of strips*. These compositions, introduced by Chudnovsky and Seymour [CS05], are essential to the structure of both quasi-line graphs and claw-free graphs. They generalize augmentations, but more importantly they provide a way to build many claw-free graphs using a beautiful generalization of line graphs.

Consider a multigraph $H$, possibly containing loops. To find its line graph $L(H)$ we begin with a vertex $u_e$ for each edge of $H$. For every $v \in V(H)$ we define the set $C_v \subseteq V(L(H))$ as

$$C_v = \{u_e \mid e \text{ is incident to } v \text{ in } H\}.$$

Two vertices $u$ and $u'$ in $L(H)$ are adjacent if and only if they are both in $C_v$ for some $v \in V(H)$. We construct $L(H)$ from our set $\{u_e \mid e \in E(H)\}$ of isolated vertices by adding edges to make each $C_v$ a clique.

This perspective invites a generalization of line graphs. Instead of replacing each edge in $H$ with a vertex, we will replace each edge with a claw-free graph, being careful not to create a claw in the process.

**Definition 5.8.** *A* strip $(S, X, Y)$ *is a claw-free graph $S$ with two cliques $X$ and $Y$ such that for any vertex $v \in X$ (resp. $Y$), the neighbourhood of $v$ outside $X$ (resp. $Y$) is a clique.*

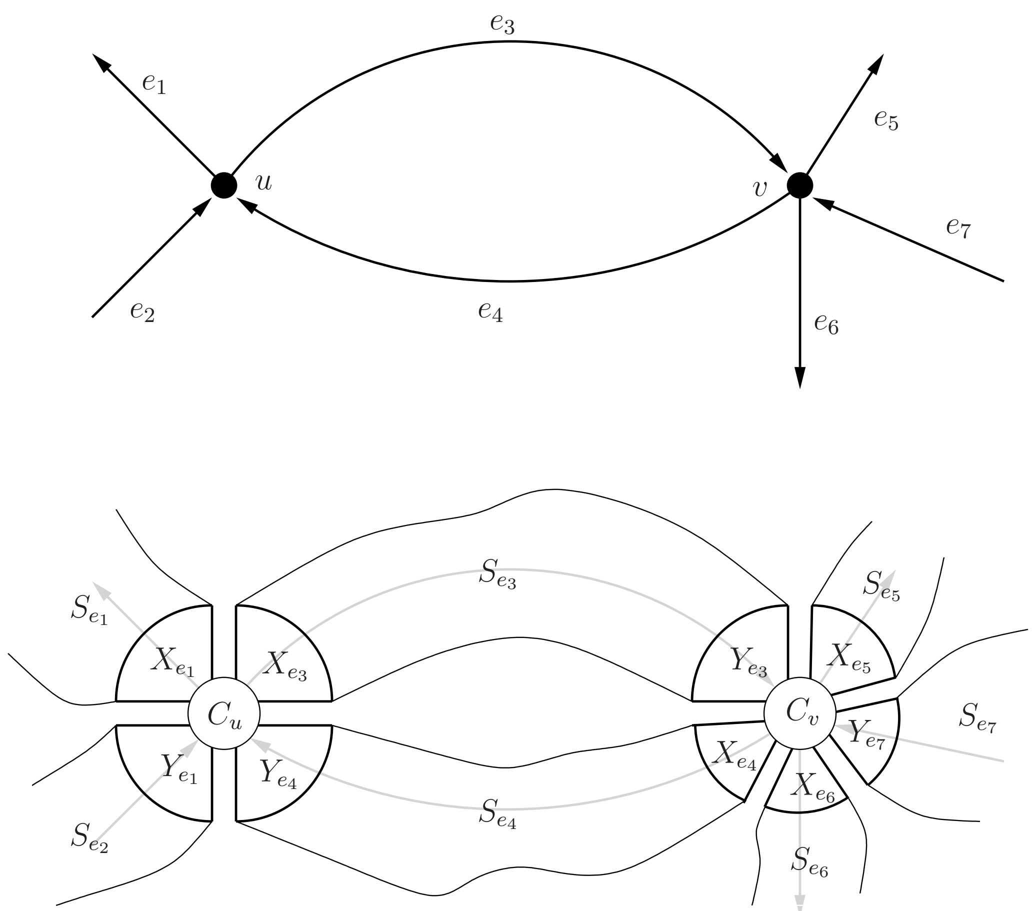


Figure 5.3: We compose a set of strips $\{(S_e, X_e, Y_e) \mid e \in E(H)\}$ by joining them together on their end-cliques. A hub clique $C_u$ will arise for each vertex $u \in V(H)$.

We compose $m$ strips in the following way. First we take a directed multigraph $H$ with $m$ edges corresponding to the $m$ strips. That is, for every every edge $e$ of $H$ we have a strip $(S_e, X_e, Y_e)$. For $v \in V(H)$ we define the set $C_v$ as

$$C_v = \left(\bigcup\{X_e \mid e \text{ is an edge out of } v\}\right) \cup \left(\bigcup\{Y_e \mid e \text{ is an edge into } v\}\right).$$

We construct $G$ from the disjoint union of $\{S_e \mid e \in E(H)\}$ by making each $C_v$ a clique (see Figure 5.3). If $G$ can be built in such a way, we say that $G$ is a *composition of strips*, and that the multigraph $H$ is the *underlying multigraph*. We direct the edges of $H$ to avoid ambiguity between $X_e$ and $Y_e$: $X_e$ corresponds to the tail of $e$, and $Y_e$ corresponds to the head.

Note that $X_e$ and $Y_e$ need not be disjoint. If $X_e = Y_e = S_e$ for every strip then $G$ is the line graph of the multigraph obtained by replacing each edge $e$ of $H$ with $|S_e|$ copies of it. Observe that any vertex in $X_e$ or $Y_e$ will be bisimplicial in $G$, and as a result we can be sure that $G$ is claw-free. Furthermore, $G$ is quasi-line precisely when each $S_e$ is quasi-line. Compositions of strips clearly generalize line graphs, and they also extend augmentations of line graphs. If we have a strip

$(S, X, Y)$ in which $X$ and $Y$ partition the vertices of $S$, then not only is $(X, Y)$ a homogeneous pair of cliques, but $G$ is an augmentation of the graph $G'$ that we get by replacing $X$ and $Y$ with two adjacent vertices. Conversely, every augmentation of a line graph $H$ is a composition of strips for which $H$ is the underlying line graph.

**Remark:** Chudnovsky and Seymour originally defined a strip $(S', a, b)$ as a claw-free graph in which $a$ and $b$ are nonadjacent simplicial vertices [CS05, KR08b]. We define $(S, X, Y)$ where $S = S' - \{a, b\}$, $X = N(a)$, and $Y = N(b)$. Our definition is basically equivalent and is easier to work with.

### 5.2.1 2-joins arising from compositions

To solve optimization problems on compositions of strips we need to know how to decompose them. We will do this via *2-joins*. Our definition is slightly different than the definition used by Cornuéjols and Cunningham [CC85].

**Definition 5.9.** *Suppose vertex sets $V_1$ and $V_2$ partition $V(G)$ and there are cliques $X_i$ and $Y_i$ in $V_i$ such that $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are cliques, and there are no other edges between $V_1$ and $V_2$. Then we say that $((X_1, Y_1), (X_2, Y_2))$ is a* 2-join.

This is equivalent to a *generalized 2-join* as defined by Chudnovsky and Seymour [CS05].

In a composition of strips we replace each edge $xy$ of the underlying multigraph by a strip $S_{xy}$ and attach it to the graph by joining $X_{xy}$ to $C_x$ and joining $Y_{xy}$ to $C_y$. Thus it is clear how we should decompose these graphs, because we end up with a 2-join for any strip $S_{xy}$. The 2-join is

$$(\ (C_x \setminus X_{xy},\ C_y \setminus Y_{xy}),\ (X_{xy},\ Y_{xy})\ ).$$

Knowing that a claw-free graph admits such a 2-join is not enough to prove our bounds on the chromatic number. We need to know the structure of $S_{xy}$ so we can exploit properties of restricted colourings in the strips. It turns out that we need five types of strips to describe claw-free graphs, but only one to describe quasi-line graphs.

## 5.3 The structure of quasi-line graphs

### 5.3.1 The basics

Chudnovsky and Seymour showed that any quasi-line graph containing no homogeneous pair of cliques is either a circular interval graph or a composition of one special type of strip [CS05], which we define now.

**Definition 5.10.** *Let $S$ be a linear interval graph with vertices $v_1, \ldots, v_n$ in linear order, and let $X = \{v_1, \ldots, v_{|X|}\}$ and $Y\{v_{n-|Y|+1}, \ldots, v_n\}$ be cliques in $S$. Then $(S, X, Y)$ is a* linear interval strip.

Chudnovsky and Seymour [CS05] proved:

**Theorem 5.11.** *Every quasi-line graph containing no homogeneous pair of cliques is a circular interval graph or a composition of linear interval strips.*

To decompose quasi-line graphs we will exploit the structure of the 2-joins corresponding to linear interval graphs. A 2-join $((X_1, Y_1), (X_2, Y_2))$ separating $G_1$ and $G_2$ is an *interval 2-join* if $(G_2, X_2, Y_2)$ is a linear interval strip. We say that an interval 2-join is *trivial* if $V(G_2) = X_2 = Y_2$. A nontrivial interval 2-join is *canonical* if $X_2 \cap Y_2$ is empty. Given a nontrivial interval 2-join $((X_1, Y_1), (X_2, Y_2))$ where $C = X_2 \cap Y_2$, observe that $((X_1 \cup C, Y_1 \cup C), (X_2 \setminus C, Y_2 \setminus C))$ is a canonical interval 2-join.

If every interval 2-join in a composition of linear interval strips is trivial then the graph is a line graph. This implies a useful decomposition result:

**Lemma 5.12.** *For any quasi-line graph $G$, one of the following is true:*

- *$G$ contains a homogeneous pair of cliques*
- *$G$ is a circular interval graph*
- *$G$ is a line graph*
- *$G$ admits a canonical interval 2-join.*

Since we already know a lot about colouring circular interval graphs and line graphs, to prove Theorem III.1 and Theorem III.2 our only concern is how to deal with homogeneous pairs of cliques and canonical interval 2-joins. We now introduce *thickenings*, which generalize augmentations. They help us to understand homogeneous pairs of cliques in quasi-line and claw-free graphs.

### 5.3.2 Thickenings

Here we present *thickenings*, which broaden the definition of augmentations in two ways. First we allow vertex multiplication, so we can focus on a description of graphs containing no homogeneous clique. Both quasi-line and claw-free graphs are closed under vertex multiplication. Second, we modify the restrictions on the matching $M$, allowing it to be empty or to contain any edge whose removal does not introduce a claw. This relaxes the restriction we use for augmentations, i.e. that no edge in $M$ is in a triangle.

**Definition 5.13.** *An edge $e$ in a claw-free graph $G$ is* claw-neutral *if its removal does not introduce a claw. A matching $M$ in $G$ is* claw-neutral *if every edge in $M$ is claw-neutral.*

**Observation 5.14.** *If $M$ is a claw-neutral matching in a claw-free graph $G$, then $G - M$ is claw-free.*

Let $M$ be a claw-neutral matching in a claw-free graph $G$. We say that $G'$ is a *thickening of $G$ under $M$* (or sometimes just a thickening of $G$) if we can construct it from $G$ in the following way. First we multiply each vertex, substituting a nonempty clique $I(v)$ for every vertex $v$ of $G$. Then for every $uv \in M$, we remove from $G'$ a nonempty proper subset of the edges between $I(u)$ and $I(v)$. If $M = \emptyset$ we say that $G'$ is a *proper thickening* of $G$. In this case $G'$ arises from $G$ by a sequence of vertex replications. For a set $S \subseteq V(G)$ we use $I(S)$ to denote $\cup_{v \in S} I(v)$.

We have given theorems describing Berge quasi-line graphs and general quasi-line graphs containing no homogeneous pair of cliques. But these results don't really tell us anything about homogeneous pairs of cliques in a quasi-line graph. Thickenings, on the other hand, tell us how homogeneous pairs of cliques arise: some trivial pairs will arise from vertex replication, and some more interesting pairs will arise from edges in $M$. If we can describe $M$, we reach a more useful description of graphs that can be expressed as a thickening of $G$. This becomes very important when we consider claw-free graphs in later chapters.

To illustrate this point, take the example of peculiar graphs. Looking at Figure 5.1, we can see that there are three special homogeneous pairs of cliques: $(A_1, B_2)$, $(A_2, B_3)$, and $(A_3, B_1)$. Furthermore, any peculiar graph is a thickening of a circular interval graph $G$ on nine vertices under some matching $M$. The vertices of $G$ are $\{a_i, b_i, k_i \mid i \in \{1, 2, 3\}\}$, corresponding to the cliques $\{A_i, B_i, K_i \mid i \in \{1, 2, 3\}\}$. If two cliques are complete to each other, their corresponding vertices are adjacent. If they are anticomplete, their corresponding vertices are nonadjacent. If they are neither complete nor anticomplete then their corresponding vertices are adjacent, and the edge between them is in $M$. This description is more specific than the simple statement that a peculiar graph contains a homogeneous pair of cliques or is some circular interval graph on nine vertices. Obviously we do not need this specificity to colour peculiar graphs – they are perfect. But for other classes of claw-free graphs we need a good understanding of the structure of $M$. Thus the same idea will be very useful when we colour other classes of claw-free graphs in Chapter 10.

Thickenings were inspired by augmentations, and they arise naturally in the study of *trigraphs*, first introduced in Chudnovsky's Ph.D. thesis [Chu03]. Chudnovsky and Seymour actually characterized the structure of *claw-free trigraphs* [CS08b], in which a claw-neutral matching $M$ of "semi-adjacent" vertices is specified.

### 5.3.3 Fuzzy linear and circular interval graphs

Thickenings allow us to decribe two important types of quasi-line graphs. If a graph is a thickening of a linear interval graph, we say it is a *fuzzy linear interval graph*. If it is a thickening of a circular interval graph, we say it is a *fuzzy circular interval graph*.

Now suppose we have a linear interval strip $(S, X, Y)$, and $S'$ is a thickening of $S$. Then defining $X' = \cup\{I(v) \mid v \in X\}$ and $Y' = \cup\{I(v) \mid v \in Y\}$, $(S', X', Y')$ is a strip, and we say it is a *fuzzy linear interval strip*. This gives us a full description of quasi-line graphs due to Chudnovsky and Seymour [CS05]:

**Theorem 5.15.** *Every quasi-line graph is a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

To colour a quasi-line graph we must restrict the structure of its homogeneous cliques. In the next chapter we introduce two appropriate restrictions. The first is sufficient for quasi-line graphs. The second one is stronger, and we make use of it when looking at other classes of claw-free graphs.

# Chapter 6

# Skeletal Graphs

In the previous chapter we introduced thickenings, which result in homogeneous pairs of cliques. In this chapter we consider the structure of these homogeneous pairs. We show that certain types of homogeneous pairs of cliques cannot appear in a minimum counterexample to our theorems bounding $\chi$. The types of homogeneous pairs of cliques that we cannot easily eliminate from a minimum counterexample are *skeletal* homogeneous pairs of cliques, and a graph containing no nonskeletal homogeneous pair of cliques is a *skeletal* graph. As we will show, to prove our bounds on $\chi$ we need only prove them for skeletal graphs.

Skeletal quasi-line graphs and skeletal claw-free graphs can be characterized with simpler structure theorems than their unrestricted counterparts. This is the first reason we prefer to deal with them. For quasi-line graphs under a weaker restriction on homogeneous pairs of cliques, a similar result was proved by Chudnovsky and Seymour [CS05] and then applied by Chudnovsky and Fradkin [CO07] to bound the chromatic number of quasi-line graphs. Our stronger restriction simplifies the structure of claw-free graphs, not just quasi-line graphs. The second advantage of insisting that all homogeneous pairs of cliques are skeletal is that skeletal homogeneous pairs of cliques have a very simple structure. This makes it easy to analyze how invariants drop when we remove a stable set from a minimum counterexample when proving the Main Conjecture for quasi-line graphs and claw-free graphs.

## 6.1 Restricting homogeneous pairs of cliques

Given a homogeneous pair of cliques $(A, B)$ in a graph $G$ we want to remove edges between $A$ and $B$ in $G$ to reach a graph $G'$ such that:

- $G'$ is easier to describe and colour than $G$
- given a $k$-colouring of $G'$ we can easily find a $k$-colouring of $G$.

We call this action *reducing on* $(A, B)$. The fact that we can do this rests on our understanding of cobipartite graphs. In a proper colouring of $G$, the number of colours intersecting both $A$ and $B$ can be any value between 0 and $|A| + |B| - \omega(G[A \cup B])$. Now suppose we construct $G'$ from $G$ by removing edges between $A$ and $B$ without changing $\omega(G[A \cup B])$. Then since $(A, B)$ is a homogeneous pair of cliques in both $G$ and $G'$, it follows that $\chi(G) = \chi(G')$. We will prove this and give a more detailed explanation in the next section. In this section we will introduce two useful ways to restrict homogeneous pairs of cliques.

Note that one of the most natural ways to eliminate a homogeneous pair of cliques $(A, B)$ is to contract $A$ and $B$ down to single vertices. This is essentially the inverse of the thickening operation, and through repeated application allows us to arrive at a graph containing no homogeneous pair of cliques. However, we lose too much information about colourings when we do this. Specifically, suppose we want to $k$-colour $G$ and we contract $(A, B)$ down to two vertices to reach $G'$. A $k$-colouring of $G'$ doesn't tell us how we might $k$-colour $G$, or even whether such a colouring exists. Thus we must use more careful reductions on homogeneous pairs of cliques.

### 6.1.1 Linear and nonlinear homogeneous pairs of cliques

As we have seen, circular interval graphs and linear interval graphs are fundamental to the structure of quasi-line graphs. Thus one natural restriction on $(A, B)$ is to insist that $G[A \cup B]$ is a linear interval graph.

**Definition 6.1.** *A homogeneous pair of cliques $(A, B)$ is a* linear homogeneous pair of cliques *if it induces a linear interval graph. Otherwise it is called a* nonlinear homogeneous pair of cliques*.*

We can characterize linear homogeneous pairs of cliques in another way:

**Lemma 6.2.** *A homogeneous pair of cliques $(A, B)$ in a graph $G$ is nonlinear precisely if $G[A \cup B]$ contains an induced $C_4$.*

*Proof.* It is easy to see that if $G[A \cup B]$ is a linear interval graph then it contains no induced $C_4$. Now suppose $G[A \cup B]$ contains no induced $C_4$. Order the vertices $\{a_1, \ldots, a_{|A|}\}$ of $A$ and then $\{b_1, \ldots, b_{|B|}\}$ from left to right so that

- For $1 \leq i < j \leq |A|$, $a_j$ has at least as many neighbours in $B$ as $a_i$ does.
- For $1 \leq i < j \leq |B|$, $b_i$ has at least as many neighbours in $A$ as $b_j$ does.

Suppose for some $1 \leq i < j \leq |A|$, $N(a_j) \cap B$ does not contain $N(a_i) \cap B$. Then there must be vertices $b_k$ and $b_l$ such that $a_i$ sees $b_k$ but not $b_l$, and $a_j$ sees $b_l$ but not $b_k$, contradicting the fact that $G[A \cup B]$ contains no induced $C_4$. Therefore for $1 \leq i < j \leq |A|$, $N(a_j) \cap B$ contains $N(a_i) \cap B$. By symmetry, for $1 \leq i < j \leq |B|$, $N(b_i) \cap A$ contains $N(b_j) \cap A$. It follows that $G[A \cup B]$ is a linear interval graph with vertex ordering $\{a_1, \ldots, a_{|A|}, b_1, \ldots, b_{|B|}\}$. $\square$

Linear homogeneous pairs of cliques are extremely useful in characterizing the structure of quasi-line graphs. Chudnovsky and Seymour[1] [CS05] proved the following.

**Lemma 6.3.** *Any fuzzy linear (resp. fuzzy circular) interval graph containing no nonlinear homogeneous pair of cliques is a linear (resp. circular) interval graph.*

**Theorem 6.4.** *Let $G$ be a quasi-line graph containing no nonlinear homogeneous pair of cliques. Then $G$ is a circular interval graph or a composition of linear interval strips.*

Chudnovsky and Fradkin later proved that nonliner homogeneous pairs of cliques can be reduced nicely with respect to colourings [CO07].

**Lemma 6.5.** *Let $G$ be a quasi-line graph containing a nonlinear homogeneous pair of cliques. Then there is a proper subgraph $G'$ of $G$ such that $G'$ is quasi-line and $\chi(G) = \chi(G')$.*

[1] Chudnovsky and Seymour originally called linear homogeneous pair of cliques *trivial* homogeneous pairs of cliques.

The proof of this lemma rests on the simple fact that if $(A, B)$ contains a $C_4$ then we can remove an edge between $A$ and $B$ without changing the clique number of $G[A \cup B]$. We now introduce a stronger restriction on homogeneous pairs of cliques, which carries this idea through to its logical conclusion.

### 6.1.2 Skeletal homogeneous pairs of cliques

We can reduce on a nonlinear homogeneous pair of cliques $(A, B)$ by removing edges between $A$ and $B$ in an induced $C_4$ in $G[A \cup B]$ until no such edge remains. This operation makes $G[A \cup B]$ a linear interval graph and does not change the clique number of $G[A \cup B]$. If we focus on this clique number instead of induced 4-holes, then we can remove even more edges and further simplify the structure of $G[A \cup B]$.

**Definition 6.6.** *Let $(A, B)$ be a homogeneous pair of cliques in a graph $G$. We say that $(A, B)$ is* skeletal *if we cannot remove an edge between $A$ and $B$ without reducing $\omega(G[A \cup B])$.*

If $(A, B)$ is skeletal then the edges between $A$ and $B$ are contained in a single clique $\Omega(A, B)$, which we consider to be empty if there are no edges between $A$ and $B$ (see Figure 6.1). Thus $A \cup B$ is partitioned into the four sets $A \cap \Omega(A, B)$, $B \cap \Omega(A, B)$, $A \setminus \Omega(A, B)$, $B \setminus \Omega(A, B)$, each of which is a homogeneous clique, a singleton, or empty. This is the simple structure we want, and we will exploit it repeatedly when colouring claw-free graphs in Chapter 10. We want to consider graphs in which every homogeneous pair of cliques is skeletal.

**Definition 6.7.** *A graph is* skeletal *if it contains no nonskeletal homogeneous pair of cliques.*

Observe that any skeletal homogeneous pair of cliques is linear. Thus no skeletal graph contains a nonlinear homogeneous pair of cliques. This immediately gives us a characterization of skeletal quasi-line graphs as a corollary to Theorem 6.4.

**Theorem 6.8.** *Every skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.*

We use the corresponding decomposition theorem to prove our bounds on the chromatic number in the next chapter.

**Theorem 6.9.** *Any quasi-line graph that is not a line graph or a circular interval graph contains a nonskeletal homogeneous pair of cliques or admits a canonical interval 2-join.*

Insisting that the graph is skeletal does not refine the structure theorem. However, this is an anomalous case. For many classes of claw-free graphs, skeletal homogeneous pairs of cliques allow us to simplify structural characterizations in a way that linear homogeneous pairs of cliques do not. We give a simple example now, using a class that we have already seen.

#### Skeletal Berge quasi-line graphs

To illustrate the usefulness of skeletal graphs in simplifying structure theorems, we rephrase the structure theorem for Berge quasi-line graphs one last time.

**Theorem 6.10.** *Let $G$ be a skeletal Berge claw-free graph containing no clique cutset. Then $G$ is a circular interval graph or the line graph of a bipartite multigraph.*

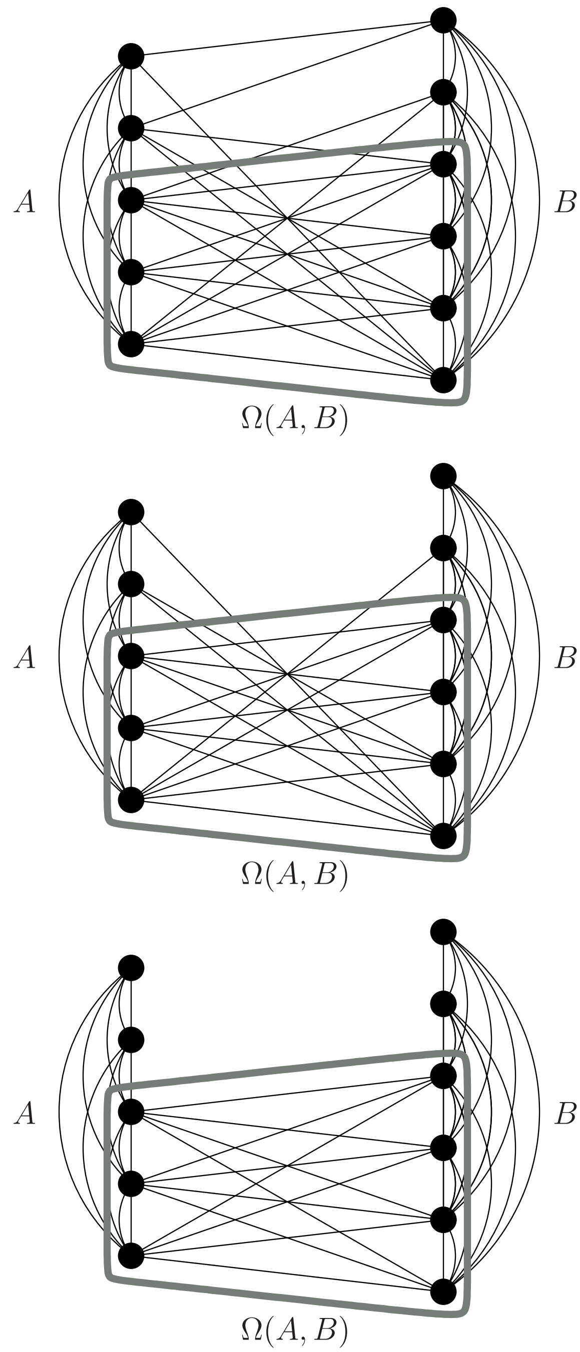


Figure 6.1: Three homogeneous pairs of cliques: one nonlinear (top), one nonskeletal linear (middle), and one skeletal (bottom). We reduce a nonskeletal homogeneous pair of cliques $(A, B)$ by removing edges without changing the size of a maximum clique in $G[A \cup B]$.

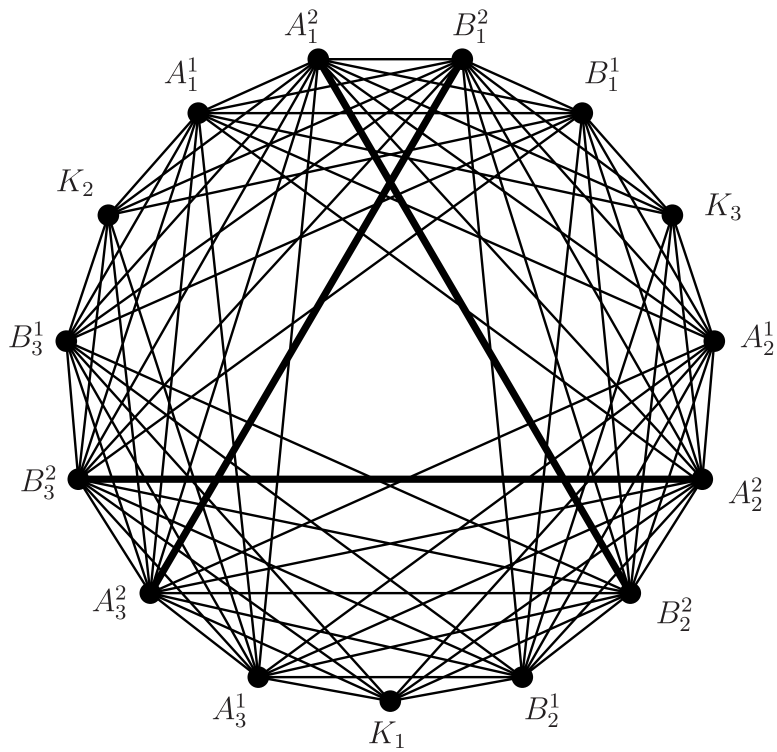


Figure 6.2: A skeletal peculiar graph. Each circle represents a clique; two cliques are complete to each other if they have an edge between them in the figure, otherwise they are anticomplete.

*Proof.* First suppose that $G$ is a skeletal peculiar graph. We claim that $G$ is a proper thickening of the circular interval graph on at most fifteen vertices shown in Figure 6.2. We split $A_i$ and $B_i$ into two cliques each for $i \in \{1, 2, 3\}$. Let $A_i^1$ (resp. $B_{i+1}^1$) be the set of vertices in $A_i$ (resp. $B_{i+1}$) with no neighbour in $B_{i+1}$ (resp. $A_i$), with indices taken modulo 3. Let $A_i^2$ and $B_i^2$ be $A_i \setminus A_i^1$ and $B_i \setminus B_i^1$ respectively. So in the skeletal homogeneous pair of cliques $(A_i, B_{i+1})$, the sets $A_i^2$ and $B_{i+1}^2$ are $\Omega(A_i, B_{i+1}) \cap A_i$ and $\Omega(A_i, B_{i+1}) \cap B_{i+1}$ respectively. When $|A_i \cup B_{i+1}| = 2$, $(A_i, B_{i+1})$ is not technically a homogeneous pair of cliques, but in this case our abuse of terminology is harmless.

Now suppose $G$ is a skeletal augmentation of a bipartite line graph $H$. Take an edge $xy$ along which $H$ was augmented by $(X, Y)$ with $|X \cup Y| > 2$. Then $(X, Y)$ is a skeletal homogeneous pair of cliques. Therefore if $X \cup Y$ is not a clique we can assume there is a vertex $v \in X$ with no neighbour in $Y$. But $N_H(x) - y$ is a clique, so $v$ is simplicial, contradicting the assumption that $G$ contains no clique cutset. Therefore $G$ is a proper thickening of a line graph of a bipartite graph, so $G$ is a line graph of a bipartite multigraph. □

Note that in this case we cannot replace "skeletal" with "containing no nonlinear homogeneous pair of cliques."

## 6.2 Reducing nonskeletal graphs

Now that we have explained how to restrict homogeneous pairs of cliques, we will prove that we can reduce nonskeletal graphs efficiently. The main goal of this section is to prove the following:

**Theorem 6.11.** *Let $G$ be a nonskeletal graph. Then there is a skeletal subgraph $G'$ of $G$ such that:*

1. *If $G$ is quasi-line (resp. claw-free) then $G'$ is also quasi-line (resp. claw-free).*
2. *If $G$ is Berge then $G'$ is also Berge.*
3. *$\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.*

*Furthermore we can find $G'$ in $O(n^2m^2)$ time, and given a $k$-colouring of $G'$ we can construct a $k$-colouring of $G$ in $O(n^2m^2)$ time.*

Crucial to the proof of this result are the following two lemmas, which we apply repeatedly:

**Lemma 6.12.** *Given a graph $G$ and a nonskeletal homogeneous pair of cliques $(A, B)$, in $O(n^{5/2})$ time we can find a proper subgraph $G'$ such that:*

1. *If $G$ is quasi-line (resp. claw-free) then $G'$ is also quasi-line (resp. claw-free).*
2. *If $G$ is Berge then $G'$ is also Berge.*
3. *$\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.*

*Furthermore given a $k$-colouring of $G'$ we can construct a $k$-colouring of $G$ in $O(n^{5/2})$ time.*

**Lemma 6.13.** *For any graph $G$, we can find a nonskeletal homogeneous pair of cliques, or determine that none exists, in $O(n^2m)$ time.*

Theorem 6.11 follows immediately from these two lemmas, because every time we reduce on a homogeneous pair of cliques we remove an edge, thus we do at most $O(m)$ reductions. We simply reduce on nonskeletal homogeneous pairs of cliques one at a time until none remains.

Reducing on a pair $(A, B)$ (i.e. applying Lemma 6.12) involves finding a maximum clique $X$ in $\omega(G[A \cup B])$ and removing all other edges between $A$ and $B$. If $|X| = \max\{|A|, |B|\}$ then we simply remove all edges between $A$ and $B$. Thus $(A, B)$ will become a skeletal homogeneous pair of cliques and $X$ will become $\Omega(A, B)$. See Figure 6.1.

These results rely heavily on the fact that any homogeneous pair of cliques $(A, B)$ induces a cobipartite (and therefore perfect) subgraph $G[A \cup B]$. Results of Hopcroft and Karp provide an $O(n^{5/2})$-time algorithm for optimally colouring a cobipartite graph and finding a maximum clique in a cobipartite graph [HK73]. Recall that if $\alpha(G) = 2$ then an optimal colouring of $G$ corresponds to a maximum matching in $\overline{G}$.

### 6.2.1 Reducing on a nonskeletal homogeneous pair of cliques

We now prove Lemma 6.12. This will tell us exactly how we reduce on a nonskeletal homogeneous pair of cliques $(A, B)$ and how we can manipulate colourings on $(A, B)$.

*Proof of Lemma 6.12.* Assume $|A| \geq |B|$. We can find a maximum clique $X$ of $G[A \cup B]$ in $O(n^{5/2})$ time, choosing $X$ to be $A$ if $A$ is a maximum clique. To construct $G'$ from $G$, we remove precisely the edges between $A$ and $B$ that are not in $X$. Clearly $\omega(G'[A \cup B]) = \omega(G[A \cup B]) = |X|$. Since $(A, B)$ is not skeletal, $G'$ is a proper subgraph of $G$. We can find $G'$ in $O(n^{5/2})$ time.

We must prove that $G'$ is claw-free. Suppose there is a vertex $v$ seeing three mutually nonadjacent vertices $a, b, c$ in $G'$. Then without loss of generality, $a \in A$, $b \in B$, and $c \notin A \cup B$ since $G$ is claw-free. Since $c$ sees neither $a$ nor $b$ in $G'$, $c$ sees nothing in $A \cup B$ in $G$. It follows that $v \notin A \cup B$, so $v$ sees all of $A \cup B$ in $G$. Therefore since $A$ and $B$ are not complete to each other in $G$, $G$ contains a claw centred at $v$, a contradiction. So $G'$ is claw-free.

Now suppose $G$ is quasi-line; we must show that $G'$ is quasi-line. Suppose a vertex $v$ is not bisimplicial in $G'$ and let $(S, T)$ be a partitioning of $N_G(v)$ into two cliques. If $v$ has a neighbour $w \in S \setminus (A \cup B)$ that sees $A$ but not $B$, then $B \subseteq T$ and thus $S \cup A$ and $T \setminus A$ are two cliques covering $N_{G'}(v)$ in $G'$. By symmetry we can assume that if no such $w$ exists then all of $N_{G'}(v) \setminus (A \cup B)$ sees $A \cup B$, therefore $(S \cup A) \setminus B$ and $(T \cup B) \setminus A$ are two cliques covering $N_{G'}(v)$ in $G'$. Therefore $G'$ is quasi-line if $G$ is quasi-line. This proves (1).

Suppose $G$ is Berge but $G'$ contains an odd hole or antihole $H$. $H$ does not contain a homogeneous set, so it must intersect each of $A$ and $B$ exactly once, in vertices $a_1$ and $b_1$. But there are nonadjacent $a_i$ and $b_j$ in $G$. Replacing $a_1$ and $b_1$ with $a_i$ and $b_i$ in $H$ gives us an odd hole or antihole in $G$, contradicting the assumption that $G$ is Berge. This proves (2).

Let $c_{G'}$ be a proper colouring of $G'$ using $k \geq \chi(G')$ colours. Since $(A, B)$ is a homogeneous pair, to construct a $k$-colouring of $G$ it is enough to find a colouring of $G[A \cup B]$ that uses the same set of colours as $c_{G'}$ on $A$ and on $B$. We can do this in $O(n^{5/2})$ time because the number of colours which appear on both $A$ and $B$ in the colouring of $G'$ is at most the maximum size of a matching in $\overline{G}'$, which is the same as the size of a maximum matching in $\overline{G}$, i.e. $|(A \cup B) - X|$.

Since $G[A \cup B]$ is perfect, this extends to fractional colourings. Specifically, for any $l \geq \omega(G[A \cup B])$ there is a fractional $l$-colouring of $G[A \cup B]$. Suppose we have a fractional $k$-colouring of $G'$. This colouring uses weight $l \geq \omega(G[A \cup B])$ on $A \cup B$, so since $(A, B)$ is a homogeneous pair of cliques we can combine the colouring of $G' - (A \cup B) = G - (A \cup B)$ with a fractional $l$-colouring of $G[A \cup B]$ to find a fractional $k$-colouring of $G$. This proves (3). □

### 6.2.2 Finding homogeneous pairs of cliques

In Chapter 3 we mentioned Everett, Klein, and Reed's $O(mn^3)$ algorithm for finding homogeneous pairs, but we have not yet discussed finding homogeneous pairs of cliques.

We begin with an algorithm for finding a nonlinear homogeneous pair of cliques. Then we will give a more efficient algorithm for finding a nonskeletal homogeneous pair of cliques in a graph containing no nonlinear homogeneous pair of cliques.

Observe that if an edge $a_1a_2$ is contained in $A$ for some nonlinear homogeneous pair of cliques $(A, B)$, then any edge $b_1b_2$ such that $\{a_1, b_1, a_2, b_2\}$ induces a $C_4$ must be contained in $B$.

**Lemma 6.14.** *For any graph $G$ we can find a nonlinear homogeneous pair of cliques in $G$, or determine that none exists, in $O(n^2m)$ time.*

*Proof.* Since a nonlinear homogeneous pair of cliques contains an induced $C_4$, we proceed by checking, for every edge contained in an induced $C_4$, whether or not there is a homogeneous pair of cliques such that this edge is in one of the cliques. Observe that if this is the case, then the other vertices of the $C_4$ are in the other clique and hence the homogeneous pair of cliques is nonlinear.

For each edge $a_1a_2$, we first check that it appears in an induced $C_4$; this can be determined in $O(m)$ time. If it is in a $C_4$, we then iteratively grow cliques $A_i$ and $B_i$ such that if there is a homogeneous pair of cliques $(A, B)$ with $a_1, a_2 \in A$, then $A_i \subseteq A$ and $B_i \subseteq B$. As mentioned above, if $b_1b_2$ is an edge and $G[\{a_1, a_2, b_1, b_2\}]$ is a $C_4$ then $b_1$ and $b_2$ must be in $B$ and so if such an $(A, B)$ exists it is a nonlinear homogeneous pair of cliques. Let $A_0 = \{a_1, a_2\}$ and let $B_0 = \emptyset$. For $t = 1, 2, \ldots n - 2$ we do the following.

1. Search for a vertex $v$ not in $A_{t-1} \cup B_{t-1}$ that sees some but not all of $A_{t-1}$ (resp. $B_{t-1}$) – it must be in $B$ (resp. $A$), so let $B_t = B_{t-1} \cup \{v\}$ (resp. $A_t = A_{t-1} \cup \{v\}$) and increment $t$. If there is no such $v$ then $(A_{t-1}, B_{t-1})$ form a nonlinear homogeneous pair of cliques; return this fact and terminate.

2. If $A_t$ and $B_t$ are not both cliques or $t = n - 2$, terminate. $A$ and $B$ do not exist.

When building $(A, B)$ we only add a vertex to the homogeneous pair if it cannot be outside the pair, hence we never face the possibility of putting an unnecessary vertex in $A \cup B$. It follows that if our method fails there is no homogeneous pair of cliques containing $A_0$. The method is clearly polytime: we can construct $(A_t, B_t)$ from $(A_{t-1}, B_{t-1})$ in $O(m)$ time, and there are $O(m)$ possible edges to check, so the total running time is at most $O(nm^2)$. However, we can actually do this in $O(n^2m)$ time by maintaining the sets of vertices outside $(A, B)$ that see all of $A$, none of $A$, all of $B$, and none of $B$ – all others must be put in the homogeneous pair. To maintain these lists, when putting a vertex $v$ into $A$, for example, we check to see if this forces any outside vertex to be put into $B$. We can check this in $O(n)$ time whenever we put a vertex in the homogeneous pair, so we can find a minimal nonlinear homogeneous pair of cliques, or determine that there is none, in $O(n^2m)$ time. □

Now we need to find linear nonskeletal homogeneous pairs of cliques. First we prove a structural result that renders the task almost trivial.

**Lemma 6.15.** *Suppose a graph $G$ contains a nonskeletal linear homogeneous pair of cliques. Then $G$ contains three nonempty disjoint cliques $A_1$, $A_2$, $B_1$ such that*

- $|A_1| \geq |B_1|$.
- *Each of $A_1$, $A_2$, and $B_1$ is either a singleton or a homogeneous clique.*
- *$A_1 \cup A_2$ is a clique, $A_2 \cup B_1$ is a clique, and there are no edges between $A_1$ and $B_1$.*
- *$(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques.*

*Proof.* Suppose the vertices of $G[A \cup B]$ are $a_1, \ldots, a_{|A|}, b_1, \ldots, b_{|B|}$ in linear order.

By swapping the names of $A$ and $B$, we can make an important assumption without loss of generality: Either $A$ is a maximum clique in $G[A \cup B]$, or there is a maximum clique $X$ of $G[A \cup B]$ and some vertex in $B$ that sees some but not all of $X \setminus B$. If we cannot assume this, then $\omega(G[A \cup B]) > \max\{|A|, |B|\}$ and there is a unique maximum clique $X$ in $(G[A \cup B])$. Furthermore since $(G[A \cup B])$ is a linear interval graph, no vertex in $A \setminus X$ (resp. $B \setminus X$) has a neighbour in $B$ (resp. $A$), contradicting the assumption that $(A, B)$ is nonskeletal.

To construct $A_1$, $A_2$, and $B_1$ we first select two vertices $a_p$ and $a_q$ in $A$. Let $p$ be minimum such that $a_p$ is in a maximum clique $X$ of $G[A \cup B]$; note that $p = 1$ if $\omega(G[A \cup B]) = |A|$. We

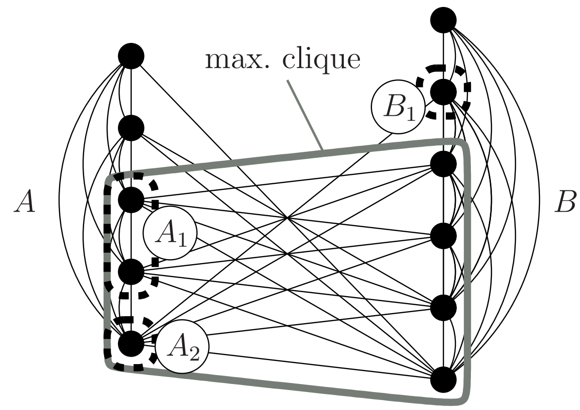


Figure 6.3: If a linear homogeneous pair of cliques is not skeletal, we can find within it a homogeneous pair of cliques with a very specific structure.

claim that there is some minimum $q > p$ such that $\bar{N}(a_p) \subset \bar{N}(a_q)$, i.e. $a_p$ and $a_q$ are not twins. If $q$ does not exist then by our above assumption either (i) $X = A$ and there are no edges between $A$ and $B$, a contradiction since $(A, B)$ is nonskeletal, or (ii) $|X| > |A|$ and no vertex in $B$ sees some but not all of $X \setminus B$, a contradiction since in this case $X$ must be the unique maximum clique of $G[A \cup B]$.

Let $A_1$ be $a_p$ along with its twins, and let $B_1$ be the set of vertices that see that see $a_p$ but not $a_q$. Clearly $B_1 \subseteq B$, and observe that $|A_1| \geq |B_1|$, otherwise $a_p$ would not be in a maximum clique in $G[A \cup B]$, whereas $a_q$ would. So let $A_2$ be $q$ along with its twins. An example is shown in Figure 6.3.

To see that $(A_1 \cup A_2, B_1)$ is a homogeneous pair of cliques, it is enough to show that $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques. By the structure of linear interval graphs, every vertex in $A \setminus (A_1 \cup A_2)$ sees either all of $B_1$ or none of $B_1$, so $B_1$ is a singleton or a homogeneous clique. Therefore $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques, following from the fact that $(A, B)$ is a homogeneous pair of cliques. Furthermore since $B_1$ is complete to $A_2$ and anticomplete to $A_1$, and $|A_1| \geq |B_1|$, it is easy to see that $(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques (in particular, $A_1 \cup A_2$ is a maximum clique in $G[A_1 \cup A_2 \cup B_1]$). □

Thus when searching for a linear nonskeletal homogeneous pair of cliques, we can focus on this specific structure.

**Lemma 6.16.** *Let $G$ be a graph containing no nonlinear homogeneous pair of cliques. Then in $O(nm)$ time we can find some nonskeletal linear homogeneous pair of cliques $(A, B)$ in $G$, or determine that $G$ is skeletal.*

Observe that Lemma 6.13 follows immediately from this lemma and Lemma 6.14.

*Proof.* We find a nonskeletal homogeneous pair of cliques $(A, B)$ by finding the cliques $A_1$, $A_2$, and $B_1$ guaranteed by the previous lemma, as follows. First we partition the vertices of $G$ into maximal homogeneous cliques in $O(m)$ time as discussed in Section 4.2.3. After that we just need to find three vertices $a_1$, $a_2$, and $b_1$ inducing a path such that $a_1$ has at least as many twins as $b_1$, no vertex sees $a_1$ but not $a_2$, and $b_1$ and its twins are the only vertices that see $a_2$ but not $a_1$. We can easily do this in $O(nm)$ time by first guessing $b_1$, then deleting $b_1$ and checking for the appropriate resulting twins in $O(m)$ time. □

Finally, we remark that we can find a skeletal homogeneous pair of cliques in $O(m)$ time. First we search for twins in time $O(m)$ – twins immediately lead to a homogeneous pair of cliques if the graph has at least four vertices. But the existence of a skeletal homogeneous pair $(A, B)$ implies the existence of twins: Either $(A \cap \Omega(A, B), B \cap \Omega(A, B))$ is a homogeneous pair of cliques with all edges between them, or $(A, B)$ is a homogeneous pair of cliques with no edges between them. Either case leads to twins. With the results of this section, this implies the following:

**Theorem 6.17.** *In $O(n^2 m)$ we can find a homogeneous pair of cliques in a graph or determine that none exists.*

## 6.3 Decomposing skeletal quasi-line graphs

We conclude the chapter with an algorithm to decompose a skeletal composition of linear interval strips. Specifically, for such a graph $G$ we will give an efficient method for determining strips $(S_e, X_e, Y_e)$ and an underlying graph $H$ such that $G$ is a composition of the strips under $H$. We call this a *strip representation* of $G$. This strip representation is by no means unique, as one can easily see by considering a cycle.

### 6.3.1 Finding a canonical interval 2-join

To find a strip representation, we first need to find a canonical interval 2-join efficiently.

**Lemma 6.18.** *Let $G$ be a connected quasi-line graph. In time $O(n^2 m)$ we can find a canonical interval 2-join in $G$ or determine that none exists.*

*Proof.* Suppose a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ exists such that $X_2$ and $Y_2$ are nonempty, and $G_2$ is connected. Then there are non-simplicial vertices $x$ and $y$ in $G$ such that $G_2$ has a linear interval representation with $x$ and $y$ at the extreme left and right. We proceed by guessing $x$ and $y$, then checking to see if they yield a desired join.

Since $x$ is not simplicial both $X_1$ and $N(x) \setminus (X_1 \cup X_2)$ are nonempty. Thus $\bar{N}(x)$ has exactly two maximal cliques, namely $X_1 \cup X_2$ and $\bar{N}(x) \setminus X_1$. We can find one maximal clique greedily in linear time, and having generated one of them, $C$, we can find the other by generating a maximal clique in $\bar{N}(x)$ containing some arbitrarily chosen element of $\bar{N}(x) \setminus C$. Thus, $X_2$ is the intersection of these two maximal cliques and there are two choices for $X_1$. Similarly we can deduce $Y_2$ and two possible choices for $Y_1$. For each of these four possible choices of $(X_1, Y_1)$ we first check if we indeed have a 2-join. We check that it is an interval 2-join by adding to $G_2$ vertices $x'$ and $y'$ with neighbourhoods $X_2$ and $Y_2$ respectively along with a vertex $z$ with neighbourhood $\{x', y'\}$, then checking to see if the result is a circular interval graph. All of this can be done in $O(m)$ time, so checking every possible $x$ and $y$ takes $O(n^2 m)$ time. Furthermore, any simplicial vertex will result in a canonical interval 2-join in a trivial way, and we can find such vertices in $O(nm)$ time. □

### 6.3.2 Decomposing a composition of linear interval strips

We are almost ready to find a strip representation of $G$. We will do this by finding strips one at a time and replacing $X_e$ and $Y_e$ with nonadjacent vertices. This ensures that we will never introduce a nonlinear homogeneous pair of cliques, so we can repeatedly apply Theorem 6.4. We continue until we have a line graph, from which we determine a strip representation of $G$. However, we must address one case in which this may introduce a claw. This occurs when $S_e$ is a clique. We now show that if each $S_e$ is a clique then $G$ is a line graph.

One case of this is when $S_e = X_e \cup Y_e$ for every strip, in which case we already know $G$ is a line graph. Suppose that every $S_e$ is a clique. To show that $G$ is a line graph, take $H$ and instead of replacing each edge $e = \{x_e, y_e\}$ with a strip, add a vertex $v_e$. We deal with two cases: the case $V(S_e) = X_e \cup Y_e$ and the case $V(S_e) \setminus (X_e \cup Y_e) \neq \emptyset$. First suppose $V(S_e) = X_e \cup Y_e$. Put $|X_e \setminus Y_e|$ edges between $v_e$ and $x_e$, put $|Y_e \setminus X_e|$ edges between $v_e$ and $y_e$, and put $|X_e \cap Y_e|$ edges between $x_e$ and $y_e$. If $V(S_e)$ contains a vertex outside $X_e \cup Y_e$ but is still a clique, then by the definition of a strip we can see that $X_e$ and $Y_e$ are disjoint (otherwise adding nonadjacent vertices with neighbourhoods $X_e$ and $Y_e$ would result in a claw). Thus as in the previous case we put $|X_e|$ edges between $v_e$ and $x$, $|Y_e|$ edges between $v_e$ and $y$, but now we put $|V(S_e) \setminus (X_e \cup Y_e)|$ edges between $v_e$ and a new pendant vertex $z_e$. If we construct $H'$ from $H$ by doing this for each strip $(S_e, X_e, Y_e)$, then $G$ will be the line graph of $H'$.

Thus we will disregard canonical interval 2-joins for which $G_2$ is a clique, allowing us to proceed safely and still arrive at a line graph.

**Theorem 6.19.** *Let $G$ be a connected quasi-line graph containing no nonlinear homogeneous pair of cliques. Then in $O(n^2 m)$ time we can either determine that $G$ is a circular interval graph or find a strip representation of $G$.*

*Proof.* If $G$ is a circular interval graph, we can determine this in $O(m)$ time. If $G$ is a line graph, then in $O(m)$ time we can determine this and find a multigraph $H$ such that $G = L(H)$, as discussed in Section 4.2.3. Note that this gives us a strip representation of $G$, setting each strip to be a singleton.

So assume that $G$ is neither a line graph nor a circular interval graph. We begin by finding a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ such that $G_2$ is connected and is not a clique; we are guaranteed that this exists because $G$ is not a line graph, as we discussed above. We proceed by insisting that $G_2$ is a strip and recursing in the following way. We construct $G_S$ from $G$ by deleting $G_2$ and adding nonadjacent vertices $x_2$ and $y_2$ with neighbourhoods $X_1$ and $Y_1$ respectively. It is easy to see that $G_S$ is an induced subgraph of $G$, so it is quasi-line.

We claim that $G_S$ contains no nonlinear homogeneous pair of cliques, so by Theorem 6.4 it is a circular interval graph or a composition of linear interval strips. To see this, suppose $G_S$ contains a nonlinear homogeneous pair of cliques $(A, B)$. If neither $x_2$ nor $y_2$ is in $A \cup B$, then each of $A$ and $B$ is either in $X_1$ or $Y_1$, or completely outside $X_1 \cup Y_1$. In this case it is easy to confirm that $(A, B)$ is a nontrivial homogeneous pair of cliques in $G$, a contradiction. So assume at least one of $x_2$ and $y_2$, say $x_2$, is in $A$. Thus $A \subseteq X_1$.

Clearly in this case $B$ is not in $X_1$. If $B \cap X_1 = \emptyset$, then it is straightforward to confirm that $(A \setminus \{x_2\}, B \setminus \{y_2\})$ is a nonlinear homogeneous pair of cliques since neither $x_2$ nor $y_2$ is in an induced $C_4$ (but note that $y_2$ is not necessarily in $B$). So assume that $B$ is partially in $X_1$. Since $X_1 \cap B$ is complete to $A \cap B$, $(A, B \setminus X_1)$ is a nonlinear homogeneous pair of cliques in $G_S$. Thus so is $(A \setminus \{x_2\}, B \setminus (X_1 \cup \{y_2\})$. But as in the previous paragraph, this implies the existence of a nonlinear homogeneous pair of cliques in $G$, a contradiction. Therefore $G_S$ contains no nonlinear homogeneous pair of cliques and is either a circular interval graph or a composition of linear interval strips.

Observe that if $G_S$ is a circular interval graph, then it is also a composition of linear interval strips. Four strips, to be exact, two of which are singletons $(\{x_2\}, \{x_2\}, \{x_2\})$ and $(\{y_2\}, \{y_2\}, \{y_2\})$. Thus it follows easily that $G$ is a composition of three linear interval strips, and the underlying multigraph $H$ consists of three parallel edges. We can easily find this strip representation in $O(m)$ time by first finding a circular interval representation of $G_S$.

If $G_S$ is a composition of linear interval strips, then we find a strip representation of it recursively in $O(n^2m)$ time. If $x_2$ and $y_2$ are two singleton strips then we can proceed as before. Otherwise we can modify the strips to find a strip representation in which $x_2$ and $y_2$ are singleton strips. If $x_2$ is not in a singleton strip but is rather in an end-clique of another strip, say $X_e$, then we can simply remove $x_2$ from $X_e$ and insist that $x_2$ is a singleton strip because it is simplicial and therefore sees nothing outside its own hub clique. If $x_2$ is in the middle of a strip $S_e$, i.e. not in $X_e \cup Y_e$, then since $x_2$ is simplicial we can break $S_e$ into three smaller strips: one is $x_2$, and the other two consist of the vertices to the left and to the right of $x_2$, respectively, in a linear interval representation of $(S_e, X_e, Y_e)$. We can easily do the extra work for this step in $O(nm)$ time.

The only thing to make clear is how we find our 2-joins – we do not want to spend $O(n^2m)$ time finding one at each step. Our method for finding canonical interval 2-joins actually finds all possible canonical interval 2-joins, i.e. $O(n^2)$ of them. We must maintain our set of possible subgraphs $G_2$ – in a strip representation they will partition the graph, so we never want to take a 2-join giving us a $G_2$ that intersects a vertex already selected to be in a strip. Each time we make a recursion step, we just take the first canonical interval 2-join such that $G_2$ is connected and $X_2$ and $Y_2$ are both nonempty, and such that $G_2$ contains no vertex already assigned to be in a strip. We can do this in $O(n^2m)$ time overall, and this method naturally limits us to making $O(n)$ iterations. The result follows. $\square$

We remark that this algorithm gives us a strip decomposition with a nice property: Every strip $(S_e, X_e, Y_e)$ is either a singleton or has $S_e$ connected and not a clique, and $X_e$ and $Y_e$ disjoint. We call such a strip representation a *canonical strip representation*.

# Chapter 7

# Colouring Quasi-line Graphs

Now we can apply the tools that we have gathered to bound the chromatic number of quasi-line graphs. Given any quasi-line graph $G$ we can find a skeletal quasi-line subgraph $G'$ with equal chromatic number, so any minimum counterexample to Theorem III.1 or Theorem III.2 must be skeletal. Thus by Theorem 6.8 we need only prove these results for circular interval graphs and compositions of linear interval strips.

We begin by proving the Main Conjecture. Our approach is to decompose a minimum counterexample on a canonical interval 2-join, colour what remains, then complete the colouring of the entire graph. This is a very straightforward approach. To prove Theorem III.2 we need to be a bit more clever. As we will show, both proofs lead to polynomial-time algorithms that find a colouring satisfying their respective bounds.

## 7.1 Proving the Main Conjecture

Theorem 4.1 implies that the Main Conjecture holds for circular interval gaphs. Theorem 4.5 tells us that the Main Conjecture holds for line graphs. Therefore by Theorems 6.9 and 6.11 we only need to prove that a minimum conterexample to Theorem III.1 cannot admit a canonical interval 2-join. We do this now.

### 7.1.1 Dealing with canonical interval 2-joins

In Chapter 3 we showed that clique cutsets are easy to deal with when colouring graphs. This idea is relevant to the 2-joins arising from compositions of strips in two ways. First, if some $X_i$ or $Y_i$ is empty then our 2-join amounts to a clique cutset and $G_2$ is a linear interval graph. So if we have a $\gamma(G_1)$-colouring of $G_1$ then we can easily extend it to a $\gamma(G)$-colouring of $G$. Second, we can paste together colourings on 2-joins provided we have suitable colourings of $G_1$ and $G_2$ – this is what we did when proving that 2-joins preserve perfection in Chapter 3.

Before stating Lemma 7.1, which implies that no minimum counterexample to the Main Conjecture or Local Strengthening admits a canonical interval 2-join, we give some notation. Given $G$ admitting a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ let $H_2$ denote $G[V_2 \cup X_1 \cup Y_1]$.

For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in $H_2$ containing $v$ and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^i(H_2)$ as $\max_{v \in H_2} \lceil d(v) + 1 + \omega'(v) \rceil$ (here the superscript $i$ denotes *interval*). Observe that $\gamma_l^i(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$.

**Lemma 7.1.** *Let $G$ be a graph on $n$ vertices and suppose $G$ admits a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l \geq \gamma_l^i(H_2)$, we can find a proper $l$-colouring of $G$ in $O(nm)$ time.*

Since $\gamma_l^i(H_2) \leq \gamma_l(G) \leq \gamma(G)$ this lemma implies the Main Conjecture for quasi-line graphs. Furthermore it implies that a proof of the Local Strengthening for line graphs would imply the Local Strengthening for quasi-line graphs.

*Proof.* We proceed by induction on $l$, observing that the case $l = 1$ is trivial. We begin by modifying the colouring so that the number $k$ of colours used in both $X_1$ and $Y_1$ in the $l$-colouring of $G_1$ is maximal. That is, if a vertex $v \in X_1$ gets a colour that is not seen in $Y_1$, then every colour appearing in $Y_1$ appears in $N(v)$. This can be done in $O(n^2)$ time. If $l$ exceeds $\gamma_l^i(H_2)$ we can just remove a colour class in $G_1$ and apply induction on what remains. Thus we can assume that $l = \gamma_l^i(H_2)$ and so if we apply induction we must remove a stable set whose removal lowers both $l$ and $\gamma_l^i(H_2)$.

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the colouring of $G_1$ to an $l$-colouring of $G$ in one step. In other cases we remove a colour class in $G_1$ together with vertices in $G_2$ such that everything we remove is a stable set, and when we remove it we reduce $\gamma_l^i(v)$ for every $v \in H_2$; after doing this we apply induction on $l$. Notice that if $X_1 \cap Y_1 \neq \emptyset$ and there are edges between $X_2$ and $Y_2$ we may have a large clique in $H_2$ which contains some but not all of $X_1$ and some but not all of $Y_1$; this is not necessarily obvious but we deal with it in every applicable case.

Case 1. $Y_1 \subseteq X_1$.

$H_2$ is a circular interval graph and $X_1$ is a clique cutset. We can $\gamma_l(H_2)$-colour $H_2$ in $O(n^{3/2})$ time using Theorem 4.2. By permuting the colour classes we can ensure that this colouring agrees with the colouring of $G_1$. In this case $\gamma_l(H_2) \leq \gamma_l^i(H_2) \leq l$ so we are done. By symmetry, this covers the case in which $X_1 \subseteq Y_1$.

Case 2. $k = 0$ and $l > |X_1| + |Y_1|$.

Here $X_1$ and $Y_1$ are disjoint. Take a stable set $S$ greedily from left to right in $G_2$. By this we mean that we start with $S = \{v_1\}$, the leftmost vertex of $X_2$, and we move along the vertices of $G_2$ in linear order, adding a vertex to $S$ whenever doing so will leave $S$ a stable set. So $S$ hits $X_2$. If it hits $Y_2$, remove $S$ along with a colour class in $G_1$ not intersecting $X_1 \cup Y_1$; these vertices together make a stable set. If $v \in G_2$ it is easy to see that $\gamma_l^i(v)$ will drop: every remaining vertex in $G_2$ either loses two neighbours or is in $Y_2$, in which case $S$ intersects every maximal clique containing $v$. If $v \in X_1 \cup Y_1$ then since $X_1$ and $Y_1$ are disjoint, $\omega'(v)$ is either $|X_1| + |X_2|$ or $|Y_1| + |Y_2|$; in either case $\omega'(v)$, and therefore $\gamma_l^i(v)$, drops when $S$ and the colour class are removed. Therefore $\gamma_l^i(H_2)$ drops, and we can proceed by induction.

If $S$ does not hit $Y_2$ we remove $S$ along with a colour class from $G_1$ that hits $Y_1$ (and therefore not $X_1$). Since $S \cap Y_2 = \emptyset$ the vertices together make a stable set. Using the same argument as before we can see that removing these vertices drops both $l$ and $\gamma_l^i(H_2)$, so we can proceed by induction.

Case 3. $k = 0$ and $l = |X_1| + |Y_1|$.

Again, $X_1$ and $Y_1$ are disjoint. By maximality of $k$, every vertex in $X_1 \cup Y_1$ has at least $l - 1$ neighbours in $G_1$. Since $l = |X_1| + |Y_1|$ we know that $\omega'(X_1) \leq |X_1| + |Y_1| - |X_2|$ and

$\omega'(Y_1) \le |X_1| + |Y_1| - |Y_2|$. Thus $|Y_1| \ge 2|X_2|$ and similarly $|X_1| \ge 2|Y_2|$. Assume without loss of generality that $|Y_2| \le |X_2|$.

We first attempt to $l$-colour $H_2 - Y_1$, which we denote by $H_3$, such that every colour in $Y_2$ appears in $X_1$ – this is clearly sufficient to prove the lemma since we can permute the colour classes and paste this colouring onto the colouring of $G_1$ to get a proper $l$-colouring of $G$. If $\omega(H_3) \le l - |Y_2|$ then this is easy: we can $\omega(H_3)$-colour the vertices of $H_3$, then use $|Y_2|$ new colours to recolour $Y_2$ and $|Y_2|$ vertices of $X_1$. This is possible since $Y_2$ and $X_1$ have no edges between them.

Define $b$ as $l - \omega(H_3)$; we can assume that $b < |Y_2|$. We want an $\omega(H_3)$-colouring of $H_3$ such that at most $b$ colours appear in $Y_2$ but not $X_1$. There is some clique $C = \{v_i, \ldots, v_{i+\omega(H_3)-1}\}$ in $H_3$; this clique does not intersect $X_1$ because $|X_1 \cup X_2| \le l - \frac{1}{2}|Y_1| \le l - |Y_2| < l - b$. Denote by $v_j$ the leftmost neighbour of $v_i$. Since $\gamma_l^i(v_i) \le l$, it is clear that $v_i$ has at most $2b$ neighbours outside $C$, and since $b < |Y_2| \le \frac{1}{2}|X_1|$ we can be assured that $v_i \notin X_2$. Since $\omega(H_3) > |Y_2|$, $v_i \notin Y_2$.

We now colour $H_3$ from left to right, modulo $\omega(H_3)$. If at most $b$ colours appear in $Y_2$ but not $X_1$ then we are done, otherwise we will "roll back" the colouring, starting at $v_i$. That is, for every $p \ge i$, we modify the colouring of $H_3$ by giving $v_p$ the colour after the one that it currently has, modulo $\omega(H_3)$. Since $v_i$ has at most $2b$ neighbours behind it, we can roll back the colouring at least $\omega(H_3) - 2b - 1$ times for a total of $\omega(H_3) - 2b$ proper colourings of $H_3$.

Since $v_i \notin Y_2$ the colours on $Y_2$ will appear in order modulo $\omega(H_3)$. Thus there are $\omega(H_3)$ possible sets of colours appearing on $Y_2$, and in $2b + 1$ of them there are at most $b$ colours appearing in $Y_2$ but not $X_1$. It follows that as we roll back the colouring of $H_3$ we will find an acceptable colouring.

Henceforth we will assume that $|X_1| \ge |Y_1|$.

Case 4. $0 < k < |X_1|$.

Take a stable set $S$ in $G_2 - X_2$ greedily from left to right. If $S$ hits $Y_2$, we remove $S$ from $G$, along with a colour class from $G_1$ intersecting $X_1$ but not $Y_1$. Otherwise, we remove $S$ along with a colour class from $G_1$ intersecting both $X_1$ and $Y_1$. In either case it is a simple matter to confirm that $\gamma_l^i(v)$ drops for every $v \in H_2$ as we did in Case 2. We proceed by induction.

Case 5. $k = |Y_1| = |X_1| = 1$.

In this case $|X_1| = k = 1$. If $G_2$ is not connected then $X_1$ and $Y_1$ are both clique cutsets and we can proceed as in Case 1. If $G_2$ is connected and contains an $l$-clique, then there is some $v \in V_2$ of degree at least $l$ in the $l$-clique. Thus $\gamma_l^i(H_2) > l$, contradicting our assumption that $l \ge \gamma_l^i(H_2)$. So $\omega(G_2) < l$. We can $\omega(G_2)$-colour $G_2$ in linear time using only colours not appearing in $X_1 \cup Y_1$, thus extending the $l$-colouring of $G_1$ to a proper $l$-colouring of $G$.

Case 6. $k = |Y_1| = |X_1| > 1$.

Suppose that $k$ is not minimal. That is, suppose there is a vertex $v \in X_1 \cup Y_1$ whose closed neighbourhood does not contain all $l$ colours in the colouring of $G_1$. Then we can change the colour of $v$ and apply Case 4. So assume $k$ is minimal.

Therefore every vertex in $X_1$ has degree at least $l + |X_2| - 1$. Since $X_1 \cup X_2$ is a clique, $\gamma_l^i(H_2) \ge l \ge \frac{1}{2}(l + |X_2| + |X_1| + |X_2|)$, so $2|X_2| \le l - k$. Similarly, $2|Y_2| \le l - k$, so

$|X_2| + |Y_2| \leq l - k$. Since there are $l - k$ colours not appearing in $X_1 \cup Y_1$, we can $\omega(G_2)$-colour $G_2$, then permute the colour classes so that no colour appears in both $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Thus we can extend the $l$-colouring of $G_1$ to an $l$-colouring of $G$.

These cases cover every possibility, so we need only prove that the colouring can be found in $O(nm)$ time. If $k$ has been maximized and we apply induction, $k$ will stay maximized: every vertex in $X_1 \cup Y_1$ will have every remaining colour in its closed neighbourhood except possibly if we recolour a vertex in Case 6. In this case the overlap in what remains is $k-1$, which is the most possible since we remove a vertex from $X_1$ or $Y_1$, each of which has size $k$. Hence we only need to maximize $k$ once. We can determine which case applies in $O(m)$ time, and it is not hard to confirm that whenever we extend the colouring in one step our work can be done in $O(nm)$ time. When we apply induction, i.e. in Cases 2, 4, and possibly 6, all our work can be done in $O(m)$ time. Since $l < n$ it follows that the entire $l$-colouring can be completed in $O(nm)$ time. □

Chudnovsky and Ovetsky Fradkin [CO07] proved that $\chi(G) \leq \frac{3}{2}\omega(G)$ for any quasi-line graph $G$, extending a result of Shannon for line graphs [Sha49]. Our theorem strengthens these results, since for any quasi-line graph $\Delta(G) \leq 2\omega(G) - 1$ and consequently $\gamma(G) \leq \frac{3}{2}\omega(G)$.

## 7.2 Asymptotics of the chromatic number for quasi-line graphs

In Chapter 4 we presented two important classes of graphs for which $\chi$ and $\chi_f$ are close: line graphs and circular interval graphs. In this section we prove that $\chi$ and $\chi_f$ are close together for quasi-line graphs. We do this by combining the structure theorem for quasi-line graphs with colouring results on line graphs and circular interval graphs.

As we mentioned in Chapter 4, Kahn first proved that $\chi$ and $\chi_f$ agree asymptotically for line graphs. Sanders and Steurer followed this with a more precise result [SS05]:

**Theorem 7.2.** *Any line graph $G$ can be coloured in polynomial time using $\chi_f(G) + \sqrt{\frac{9}{2}\chi_f(G)}$ colours.*

We now extend this result, which gives a strengthening of the main result of [KR07]:

**Theorem 7.3.** *Any quasi-line graph $G$ can be coloured in polynomial time using $\chi_f(G)+3\sqrt{\chi_f(G)}$ colours.*

We first prove that $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$, leaving algorithmic considerations until later.

Let $G$ be a minimum counterexample to the theorem. Theorem 6.11 implies that $G$ must be skeletal. Theorem 4.2 tells us that $G$ cannot be a circular interval graph, and Theorem 7.2 tells us that $G$ cannot be a line graph. Therefore $G$ is some composition of linear interval strips $(S_e, X_e, Y_e)$ with underlying directed multigraph $H$, and it must admit a canonical interval 2-join.

We deal with a 2-join by emulating a fractional colouring on the corresponding linear interval strip with an integer colouring. However, we cannot deal with these 2-joins one at a time as we did in the previous section. Each time we turn a fractional colouring of a linear interval strip $(S, X, Y)$ into an integer colouring, we need to use an extra colour in case the total weight of stable sets intersecting both $X$ and $Y$ in the fractional colouring is not an integer. Thus we must deal with all 2-joins simultaneously, emulating an optimal fractional colouring as closely as possible. Our approach is as follows:

We begin by finding a near-optimal fractional colouring of $G$ in which the total weight of stable sets intersecting both $X_e$ and $Y_e$ is an integer, for every $e$. We then use this fractional colouring to construct a line graph $G'$ by contracting each strip into a carefully chosen clique. Using Theorem 7.2, we find a $(1+o(1))\chi_f(G)$-colouring of $G'$. This colouring gives us a partial colouring of $G$ on the end-cliques. Finally we complete the colouring of $G$ by emulating our near-optimal fractional colouring on by integer colourings on the strips.

We now address the specifics of this approach.

### 7.2.1 Finding a good fractional colouring

Given a fractional colouring of $G$, we use $w_e$ to denote the total weight of stable sets in the colouring that intersect both $X_e$ and $Y_e$. Our first job is to find a near-optimal fractional colouring of $G$ in which $w_e$ is an integer for every edge $e$ of the underlying multigraph $H$. We will prove:

**Lemma 7.4.** *There is a fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$-colouring of $G$ such that $w_e$ is an integer for all $e \in E(H)$.*

To find this fractional colouring we take a fractional $\chi_f(G)$-colouring and modify it on the cliques $X_e$. We can assume that for all $e \in E(G)$, $|X_e| \leq |Y_e|$ – if $|X_e| > |Y_e|$, simply change the direction of $e$ in $H$ and swap the names of $X_e$ and $Y_e$. Note that the case $X_e = \emptyset$ will result in a simplicial vertex, which cannot exist in a minimum counterexample.

We begin by proving that we don't need to modify too many cliques $X_e$ which have edges between them. To this end, we say that an edge $e$ of $H$ is *trivial* if $X_e = Y_e$. For a trivial edge $e$, $w_e$ is always equal to $|X_e|$ and is therefore an integer. For $v \in V(H)$, let $D(v)$ denote the number of nontrivial edges out of $v$. Let $D(H)$ be the maximum of $D(v)$ over all $v \in V(H)$. We bound $D(H)$ using the following result:

**Lemma 7.5.** *If $e$ is a nontrivial edge of $H$, then $|X_e| > 3\sqrt{\chi_f(v)}$.*

*Proof.* By the minimality of $G$, every vertex in $G$ must have degree at least $\chi(G) - 1 > \omega(G) + 3\sqrt{\chi(G)} - 1$.

Let the vertices of $S_e$ be $\{u_1, \ldots, u_{V(S_e)}\}$ in linear order, such that $X_e = \{u_1, \ldots, u_{|X_e|}\}$. Because $e$ is nontrivial and $|Y_e| \geq |X_e|$, the vertex $u_{|X_e|+1}$ exists. By the structure of linear interval graphs, its closed neighbourhood outside $X_e$ is a clique if $u_{|X_e|+1}$ is not in $Y_e$. If $u_{|X_e|+1} \in Y_e$ then its neighbourhood outside $X_e$ is contained in the hub clique containing $Y_e$. Therefore $\omega(G) + |X_e| \geq d(u_{|X_e|+1}) + 1 > \omega(G) + 3\sqrt{\chi(G)}$. □

This immediately gives us a bound on $D(H)$, because every hub clique $C_v$ has size at most $\omega(G) \leq \chi_f(G)$.

**Corollary 7.6.** $D(H) < \frac{1}{3}\sqrt{\chi_f(G)}$.

We use our bound on $D(H)$ to prove the existence of the desired fractional colouring. We omit the simple proof of the following lemma, which reduces to finding an optimal edge colouring of a set of disjoint stars:

**Lemma 7.7.** *We can colour the nontrivial edges of $H$ with $D(H) < \frac{1}{3}\sqrt{\omega(G)}$ colours such that no two edges out of the same vertex get the same colour.*

We deal with one of these colour classes at a time:

**Lemma 7.8.** *Let $E_1$ be a set of nontrivial edges in $H$, no two of which go out of the same vertex. Given a fractional $k$-colouring of $G$ with overlaps $\{w_e \mid e \in E\}$, there is a fractional $(k+1)$-colouring of $G$ with overlaps $\{w'_e \mid e \in E\}$ such that $w'_e = \lfloor w_e \rfloor$ for $e \in E_1$, and $w'_e = w_e$ for $e \notin E_1$.*

*Proof.* Take some single edge $e$ of $E_1$ and an optimal fractional colouring of $G$. We claim that we can make $w'_e$ an integer by adding $w_e - \lfloor w_e \rfloor$ extra weight to the fractional colouring. To do this, take a collection of stable sets, each intersecting both $X_e$ and $Y_e$, of total weight $w_e - \lfloor w_e \rfloor$ in the colouring (it may be necessary to split one stable set into two identical stable sets of lesser weight to do this). Now remove the vertex in $X_e$ from each of these stable sets, and fill the missing weight in $X_e$ (i.e. $w_e - \lfloor w_e \rfloor$) with singleton stable sets. This gives us the desired fractional colouring in which $w'_e$ is an integer. Note that we did not change the colouring outside $X_e$, so every other overlap is unchanged.

To see that we can "integralize" each $w_e$ using extra weight less than 1, note that for $e, e' \in E_1$, there are no edges between $X_e$ and $X_{e'}$. Thus we can integralize $X_e$ for every $e \in E_1$ using $|E_1|$ extra weight, and replace the resulting singleton colour classes with stable sets of size up to $|E_1|$, with total weight less than 1. □

This gives us an easy proof of Lemma 7.4:

*Proof of Lemma 7.4.* Begin with a fractional $\chi_f(G)$-colouring of $G$ and a colouring of the edges of $H$ guaranteed by Lemma 7.7. For each matching in this edge colouring, apply Lemma 7.8. The result is a fractional $(\chi_f(G) + D(H))$-colouring of $G$ for which each overlap $w_e$ is an integer. □

### 7.2.2 Completing the proof

We are now ready to prove our bound on $\chi(G)$.

*Proof of Theorem 7.3.* Let $G$ be a minimum counterexample to the theorem. We know that it is a composition of strips $(S_e, X_e, Y_e)$ with underlying multigraph $H$. We construct a line graph $G'$ from $G$ as follows. Take a fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$-colouring of $G$ such that every $w_e$ is an integer. We want to replace every strip $(S_e, X_e, Y_e)$ with a new strip $(S'_e, X'_e, Y'_e)$ such that (i) the resulting composition $G'$ of these strips is a line graph, (ii) $|X'_e| = |X_e|$ and $|Y'_e| = |Y_e|$, and (iii) in any colouring of $G'$, precisely $w_e$ colours intersect both $X'_e$ and $Y'_e$.

With this in mind, for each $e$ we define a new linear interval strip $(S'_e, X'_e, Y'_e)$ on $|X_e| + |Y_e| - w_e$ vertices such that $X'_e$ and $Y'_e$ are cliques of size $|X'_e|$ and $|Y'_e|$ respectively, and $|X'_e \cap Y'_e| = w_e$. Furthermore $S'_e$ itself is a clique. Let $G'$ be the composition of the strips $(S'_e, X'_e, Y'_e)$ with $H$ as the underlying multigraph (see Figure 7.1). Equivalently, we construct a multigraph $H'$ from $H$ by replacing every edge $e$ of $H$ with a triangle on $|X_e| + |Y_e| - w_e$ edges, exactly $w_e$ of which are between the original endpoints of $e$. We then set $G' = L(H')$.

We can see that $\chi_f(G') \le \chi_f(G) + \frac{1}{3}\sqrt{\omega(G)}$: Because $X'_e \setminus Y'_e$, $Y'_e \setminus X'_e$, and $X'_e \cap Y'_e$ are all homogeneous cliques in $G'$, we can safely map the colours in $G$ that hit $X_e$ (resp. $Y_e$) into stable sets that hit $X'_e$ (resp. $Y'_e$). Thus Theorem 7.2 gives us a bound on $\chi(G')$:

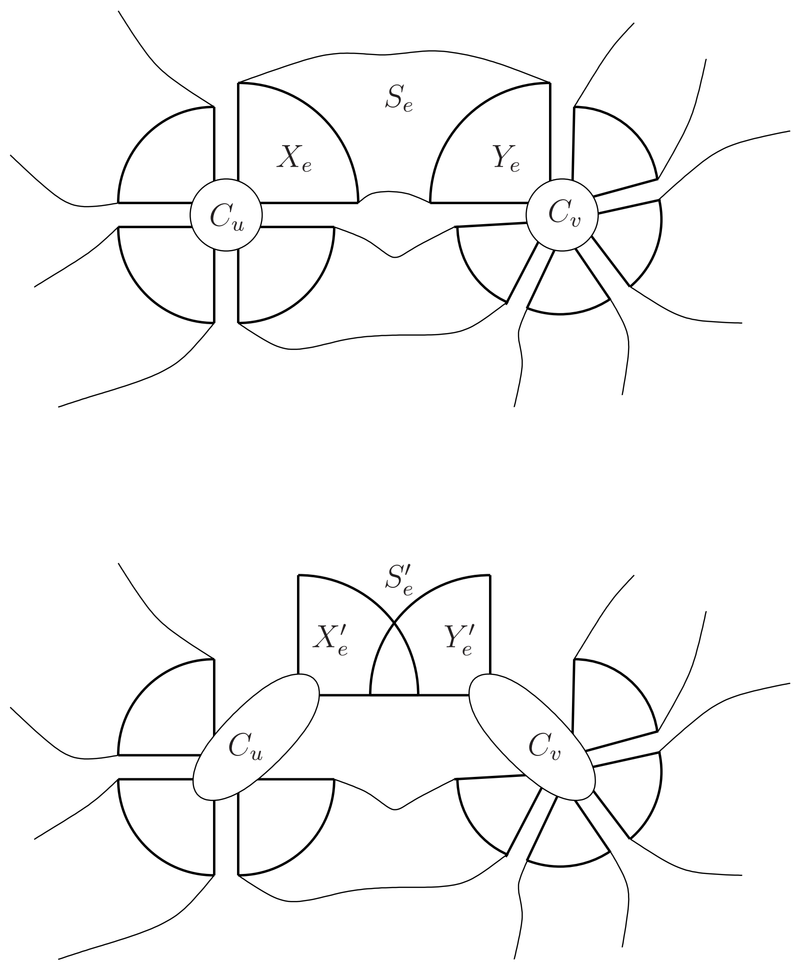


Figure 7.1: We construct a line graph $G'$ from $G$ by contracting each strip $(S_e, X_e, Y_e)$ into a $(S'_e, X'_e, Y'_e)$ where $S'_e$ is a clique, $X'_e \cup Y'_e$ together cover $S'_e$, and $|X'_e \cap Y'_e| = w_e$.

$$\begin{aligned}
\chi(G') &\leq \chi_f(G') + \sqrt{\frac{9}{2}\chi_f(G')} && (7.1)\\
&\leq \chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)} + \sqrt{\frac{9}{2}\left(\chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)}\ \right)} && (7.2)\\
&\leq \chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)} + \sqrt{\frac{9}{2}\left(\frac{4}{3}\chi_f(G)\right)} && (7.3)\\
&\leq \chi_f(G) + 3\sqrt{\chi_f(G)}. && (7.4)
\end{aligned}$$

Consider a colouring of $G'$ using $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ colours. Since $S'_e$ is a clique for every edge $e$ of $H$, the number of colours appearing in both $X'_e$ and $Y'_e$ is precisely $w_e$. So from this colouring of $G'$ we can construct a partial colouring of $G'$, colouring $\cup\{X_e \cup Y_e \mid e \in E(H)\}$ with $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ colours so that precisely $w_e$ colours appear in both $X_e$ and $Y_e$.

The original fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$-colouring of $G$ gives us a fractional colouring of each $S_e$ such that the overlap between $X_e$ and $Y_e$ is precisely $w_e$, so Lemma 4.3 tells us that we can $\lceil \chi_f(G) + \frac{1}{3}\sqrt{\omega(G)} \rceil$-colour each $S_e$ such that precisely $w_e$ colours appear in both $X_e$ and $Y_e$. Thus by permuting colour classes, we can complete the partial colouring of $G$ and find a proper $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$-colouring of $G$, proving the theorem. □

## 7.3 Algorithmic considerations

In this section we prove that given a quasi-line graph $G$, in polynomial time we can colour $G$ using $\min\{\gamma(G), \chi_f(G) + 3\sqrt{\chi_f(G)}\}$ colours.

### 7.3.1 The Main Conjecture

Our approach to colouring quasi-line graphs as in the proof of Lemma 7.1 suggests an algorithm for $\gamma(G)$-colouring these graphs in polynomial time.

**Theorem 7.9.** *Let $G$ be a quasi-line graph on $n$ vertices. Then in $O(n^2m^2)$ time we can find a proper $\gamma(G)$-colouring of $G$.*

*Proof.* We first apply Theorem 6.11 to construct a skeletal quasi-line subgraph $G'$ of $G$ such that $\chi(G) = \chi(G')$ and $\chi_f(G) = \chi_f(G')$. We can do this in $O(n^2m^2)$ time. We then $\gamma(G')$-colour $G'$. If $G'$ is a circular interval graph then we can $\gamma$-colour it in $O(n^{3/2})$ time by Theorem 4.2. If $G'$ is a line graph we can $\gamma$-colour it in $O(n^{5/2})$ time by Theorem 4.14.

Otherwise $G'$ is a composition of linear interval strips containing no nonlinear homogeneous pair of cliques, so we first find a canonical strip representation of $G'$ as described in the previous chapter. Take a non-singleton strip $(S_e, X_e, Y_e)$. We can recursively $\gamma(G')$-colour $G - S_e$ in $O(n^2m)$ time, then apply Lemma 7.1 on the canonical interval 2-join $((X_1, Y_1), (X_e, Y_e))$ associated with $(S_e, X_e, Y_e)$. This gives us a $\gamma(G')$-colouring of $G'$ in $O(n^2m^2)$ time.

The final step is to convert this into a $\gamma(G')$-colouring of $G$. But Theorem 6.11 tells us that we can do this in $O(n^2m^2)$ time, so we are done. □

### 7.3.2 Achieving the asymptotic bound

We now prove that we can $\lfloor \chi_f(G)+3\sqrt{\chi_f(G)}\rfloor$-colour a quasi-line graph $G$ in polynomial time. Just as with $\gamma$-colouring, if $G$ is not a composition of linear interval strips then we can reduce the problem in polynomial time. If $G$ is a composition of linear interval strips but the underlying directed multigraph $H$ has $D(H) \geq \frac{1}{3}\sqrt{\chi_f(G)}$, then $G$ has a vertex $v$ of degree at most $\chi_f(G)+3\sqrt{\chi_f(G)}-1$. We can reduce this case by colouring $G-v$ and giving $v$ a colour not appearing in its neighbourhood. Thus we can assume that $G$ is a composition of linear interval strips and $D(H) < \frac{1}{3}\sqrt{\chi_f(G)}$, as in the proof of Theorem 7.3.

As with the Main Conjecture, we begin by determining a canonical strip representation of $G$. Once we have constructed the line graph $G'$ from $G$ as in the proof of Theorem 7.3, it is straightforward to show that we can find a proper $\lfloor \chi_f(G)+3\sqrt{\chi_f(G)}\rfloor$-colouring of $G$ in polynomial time. To see this, first note that Theorem 7.2 tells us that we can find the desired colouring of $G'$ in polynomial time. This immediately gives us a partial colouring of $G$. The only remaining step is to find the restricted colourings of the strips. But the proof of Lemma 4.3 tells us that this can be done by colouring an auxiliary circular interval graph for each strip. We can do this in polynomial time by Theorem 4.2.

The only remaining job is to construct the line graph $G'$ from $G$.

#### Constructing the line graph

We must now show that we can indeed construct $G'$ from $G$ in polynomial time. Minty [Min80], Nakamura and Tamura [NT01], and recently Oriolo, Pietropaoli, and Stauffer [OPS08] give polynomial-time algorithms for finding a maximum-weight stable set in any claw-free graph. By polynomial equivalence results of Grötschel, Lovász, and Schrijver (see [GLS81] §7), this implies that we can compute the fractional chromatic number of any claw-free graph in polynomial time. We can deal with weights as well: Given positive integer weights $\beta(v)$ for each vertex $v$ of the graph $G$, let $G_\beta$ be the graph obtained from $G$ by substituting a clique of size $\beta(v)$ for each vertex $v$ of $G$. We can find $\chi_f(G_\beta)$ in polynomial time in terms of $|V(G)| + \log(\max_{v\in V(G)} \beta(v))$.

But in fact the equivalence results in [GLS81] tell us that for claw-free graphs, we can efficiently optimize over the feasible region of the linear program (1.3). That is, the dual of the linear program (1.2), which we can solve to compute the fractional chromatic number. Theorem 6.5.14 in [GLS93] states:

**Theorem 7.10.** *There exists an oracle-polynomial time algorithm that, for any well-described polyhedron $(P; n, \phi)$ given by a strong separation oracle and for any $c \in \mathbb{Q}^n$, either*

1. *finds a basic optimum standard solution, or*

2. *asserts that the dual problem is unbounded or has no solution.*

We refer the reader to [GLS93] for the formal definitions of a *well-described polyhedron* and an *oracle-polynomial time algorithm*. In our specific situation, this theorem states that since we can optimize over the dual feasible region (1.3), we can find a *basic solution* to (1.2) in polynomial time for any claw-free graph. A basic solution to this linear program is a fractional colouring in which the incidence vectors of the stable sets with nonzero weight are linearly independent. Thus by the dimension of the stable set polytope, there are at most $n$ nonzero stable sets in the fractional colouring.

Once we have a basic fractional colouring of $G$, it is a simple matter to compute the overlaps for the strips and construct the line graph $G'$ in polynomial time.

# Part IV

# Claw-free Graphs

We now move on from quasi-line graphs to the more general class of claw-free graphs, with the eventual goal of proving new bounds on the chromatic number.

We note that a claw-free graph $G$ is not quasi-line precisely if there is an odd antihole in the neighbourhood of some vertex. Fouquet has shown that in fact if $\alpha(G) \geq 3$ then $G$ is not quasi-line precisely if no vertex contains an induced $C_5$ in its neighbourhood, i.e. precisely if $G$ contains no $W_5$. The next chapter begins with a proof of this result. Then, by characterizing the first and second neighbourhoods of such a $C_5$, we generalize the structure theorem for quasi-line graphs to obtain a simple structure theorem for claw-free graphs with stability number at least four.

In Chapter 9 we complete the description of skeletal claw-free graphs using Chudnovsky and Seymour's characterization of all claw-free graphs. This includes a characterization of claw-free graphs with stability number at most three, as well as a more rigorous treatment of claw-free graphs with stability number at least four. This description will give us enough knowledge about the structure of claw-free graphs to prove our bounds on the chromatic number. In Chapter 10 we consider the Main Conjecture once again. We prove that it holds for all claw-free graphs, and furthermore we prove that the Local Strengthening holds for all three-cliqued claw-free graphs.

# Chapter 8

# An Invitation to the General Structure Theorems

In this chapter we examine the structure of connected claw-free graphs. First we prove a result of Fouquet, which implies that a connected claw-free graph $G$ with $\alpha(G) \geq 3$ is quasi-line precisely if it contains no $W_5$ or *5-wheel* (see Figure 8.1). We then show that when $\alpha(G) \geq 4$, every $W_5$ in the graph is separated from its non-neighbourhood by a 1-join or a 2-join. Using this result, we easily extend the structure theorem for quasi-line graphs to a structure theorem for claw-free graphs with $\alpha(G) \geq 4$. These results were obtained independently and slightly earlier by Oriolo, Pietropaoli, and Stauffer [OPS08], and should be credited to them. We obtained the result in August of 2008, while they had presented it at a conference that May.

These results can easily be obtained as a corollary of Chudnovsky and Seymour's much more detailed decomposition and structure theorems for claw-free graphs [CS07c, CS08b]. Furthermore, both we and Oriolo, Pietropaoli, and Stauffer rely on Chudnovsky and Seymour's work, since the only known proof of the structure theorem for quasi-line graphs is also as a corollary to the claw-free structure theorem. Given this fact, one might be tempted to consider these results useless.

However, this would be incorrect for two reasons. First, if a short proof of Theorem 5.11 were to be found, this would lead to a short and very natural proof of the decomposition theorem for claw-free graphs with $\alpha \geq 4$. This would be very attractive, particularly as Chudnovsky and Seymour's proof of the decomposition theorem is very technical and over eighty pages long. Second, the decomposition given here is more intuitive and easier to work with, and has been used to prove several interesting results. For example, Oriolo, Pietropaoli, and Stauffer first used it in a new algorithm for finding a maximum weight stable set in a claw-free graph [OPS08]. Galluccio, Gentile, and Ventura have used the distinction between quasi-line graphs and claw-free graphs to make progress in characterizing the stable set polytope of claw-free graphs [GGV08]. We describe their work briefly in Section 8.4.

Unfortunately, we were unable to prove the Main Conjecture for claw-free graphs using this decomposition; we needed the extra detail provided by Chudnovsky and Seymour's theorems. We present our colouring results in Chapter 10, after presenting their structure theorems in Chapter 9. It would be of interest to find a shorter, simpler proof of their result based on the decomposition we present in this chapter.

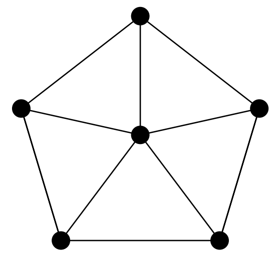

Figure 8.1: $W_5$, the smallest claw-free graph that is not quasi-line.

## 8.1 Graphs with no $C_5$ in a neighbourhood

Ben Rebea's Lemma [Ben81], proved in [CS88], states that if a claw-free graph with stability number at least three contains an odd antihole, then it contains an antihole of length 5, i.e. $C_5$. We are interested in a $C_5$ contained in the neighbourhood of some vertex. Fouquet [Fou93] provided a result analogous to Ben Rebea's Lemma; we prove it now.

**Theorem 8.1** (Fouquet)**.** *Let $G$ be a claw-free graph with $\alpha(G) \geq 3$. Then for every vertex $v$ of $G$, either $v$ is bisimplicial or the neighbourhood of $v$ contains an induced $C_5$.*

*Proof.* Assume for a contradiction that $G$ is a counterexample containing a stable set of size three. Then $G$ contains a non-bisimplicial vertex $v$ whose neighbourhood contains an odd antihole $H$ of length $k \geq 7$, and no shorter odd antihole. Denote the vertices of $H$ $v_1, \ldots, v_k$ with indices taken modulo $k$ such that $v_i$ misses $v_{i+1}$ for $1 \leq i \leq k$.

The key to the proof is to analyze the possible neighbourhoods in $H$ of the vertices in $V - H$. We begin with the following:

**Observation 8.2.** *Every vertex of $V - H$ either sees none of $H$ or misses exactly a clique of $H$.*

*Proof.* If $u \in V - H$ misses $v_i$ and $v_{i+1}$ for some $i$, then $u$ must miss $H - \{v_{i-1}, v_{i+2}\}$. Since $v_{i+2}$ is adjacent to $v_{i+4}$ and $v_{i+5}$, $u$ cannot see $v_{i+2}$ or there would be a claw. By symmetry, $u$ cannot see $v_{i-1}$, thus $u$ sees none of $H$. □

From this we easily obtain the next observation:

**Observation 8.3.** *Every vertex of $V - H$ misses exactly a clique of $H$, and hence sees more than half of the vertices of $H$.*

*Proof.* By the last observation and the fact that $\overline{H}$ induces an odd cycle, it is enough to show that no vertex misses all of $H$. Suppose the contrary. Since $G$ is connected, there are two vertices $u$ and $w$ in $V - H$ such that $u$ misses $H$ and $w$ sees some of $H$. But $w$ misses a clique of $H$, so it sees two nonadjacent vertices in $H$ and there is a claw, a contradiction. □

**Observation 8.4.** *If a vertex $u$ of $V - H$ sees $v$ then it misses at most two vertices of $H$. Furthermore if $u$ misses exactly two vertices of $H$ then they are $v_i$ and $v_{i+2}$ for some $i$.*

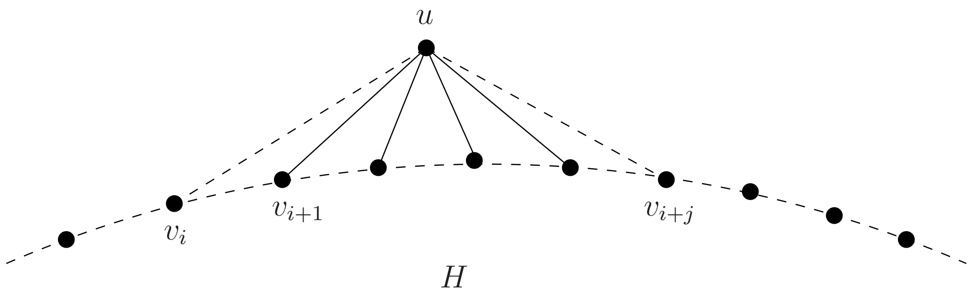


Figure 8.2: Any $u \in W$ is contained in an odd antihole of length less than $k$, passing through $u$ and a contiguous portion of $H$.

*Proof.* Suppose $u \in V - H$ sees $v$, and there exists some $i$ and some odd $j$ with $3 \leq j < k-2$ such that $u$ misses $v_i$ and $v_{i+j}$ but sees $\{v_{i+l} \mid 1 \leq l < j\}$, with indices modulo $k$. Then $\{u\} \cup \{v_{i+l} \mid 0 \leq l \leq j\}$ induces an odd antihole of length less than $k$ (see Figure 8.2), and this antihole is in the neighbourhood of $v$, contradicting the minimality of $k$.

Suppose $u$ misses exactly two vertices of $H$. Since $k$ is odd and $u$ misses no two consecutive vertices of $H$, it follows that these $i$ and $j$ exist unless $u$ misses $v_i$ and $v_{i+2}$ for some $i$: there are two paths between the non-neighbours of $u$ in $\overline{H}$, one of which is even and one of which is odd. Similarly, if $u$ misses at least three vertices of $H$ then $i$ and $j$ exist since $k$ is odd: the gaps between the non-neighbours of $u$ in $H$ have odd sum and are all at least two. □

Now let $T$ be a stable set of size three in $G$.

**Observation 8.5.** *$T$ cannot contain two vertices of $H$.*

*Proof.* This follows immediately from the fact that $\alpha(H) = 2$ and every vertex of $V - H$ misses a clique in $H$. □

**Observation 8.6.** *$T$ cannot contain exactly one vertex of $H$.*

*Proof.* Suppose $T$ contains $v_i$ and vertices $u$ and $u'$ in $V - H$. Since there is no claw in $G$, no vertex in $H - \{v_{i-1}, v_{i+1}\}$ sees both $u$ and $u'$. Thus since $k \geq 7$ and neither $u$ nor $u'$ can miss two consecutive vertices of $H$, both $u$ and $u'$ miss at least three vertices of $H$. By Observation 8.4, neither $u$ nor $u'$ sees $v$. But by Obervation 8.3 both $u$ and $u'$ see $v_{i+1}$ since they miss $v_i$. Therefore $\{u, u', v, v_{i+1}\}$ induces a claw, a contradiction. □

**Observation 8.7.** *$T$ cannot be disjoint from $H$.*

*Proof.* Suppose $T = \{t_1, t_2, t_3\}$ is contained in $V - H$. Observation 8.3 tells us that $v$ cannot be in a stable set of size three with two vertices of $T$, otherwise this stable set would be in the neighbourhood of some vertex of $H$. On the other hand $v$ cannot see all of $H$ as $G$ is claw-free. Thus $v$ sees two vertices $t_1$ and $t_2$ of $T$ and misses the third, $t_3$. By Observation 8.4, both $t_1$ and $t_2$ miss at most two vertices of $H$.

Now, no vertex of $H$ sees all of $T$ or there would be a claw. Thus $t_3$ must miss at least $k-4$ vertices of $H$, and since $t_3$ misses a clique (of size at most $\frac{k-1}{2}$) in $H$, we can see that $k = 7$ and $t_3$ misses three vertices of $H$. Thus $t_1$ and $t_2$ each miss two vertices of $H$, and since they see

$v$ Observation 8.4 tells us that there are $i_1$ and $i_2$ such that $t_1$ misses $\{v_{i_1}, v_{i_1+2}\}$ and $t_2$ misses $\{v_{i_2}, v_{i_2+2}\}$.

There is only one possible configuration. Without loss of generality we can assume that the non-neighbourhoods in $H$ of $t_1$, $t_2$, and $t_3$ are $\{v_1, v_3\}$, $\{v_5, v_7\}$, and $\{v_2, v_4, v_6\}$ respectively. Since $v$ sees $t_1$ and $t_2$, $\{t_1, v_4, t_2, v_3, v_5\}$ induces a $C_5$ in the neighbourhood of $v$, a contradiction. □

Observations 8.5, 8.6, and 8.7 together imply Theorem 8.1. □

This theorem immediately implies that if $\alpha(G) \geq 3$, $G$ is quasi-line or contains a $W_5$.

**Corollary 8.8.** *Let $G$ be a connected claw-free graph in which no neighbourhood contains an induced $C_5$. Then $\alpha(G) \leq 2$ or $G$ is quasi-line.*

We now examine how a $W_5$ and its neighbourhood can be connected to the rest of a claw-free graph.

## 8.2 Investigating the attachments of a $W_5$

In this section we will prove that if $\alpha(G) \geq 4$, then any $W_5$ and its neighbourhood are separated from the rest of the graph by a special type of 1-join or 2-join. The main results in this section were first proved by Oriolo, Pietropaoli, and Stauffer [OPS08].

### 8.2.1 The neighbourhood of a $W_5$

Let $H$ be a $C_5$ in the neighbourhood of some vertex $v$. To prove our results we partition the vertices into sets depending on their neighbourhood in $H$. We then examine how these sets interact with one another. Beginning with the vertices $v_1, \ldots, v_5$ of $H$ in cyclic order (note that in the previous section we used cyclic order in the complement) we define our sets as follows, with indices modulo 5:

- $A$ is the set of vertices seeing all of $H$; $A$ contains $v$.
- $N$ is the set of vertices with no neighbour in $H$.
- For $i = 1, \ldots, 5$, $B_i$ is the set of vertices outside $H$ whose neighbourhood in $H$ is precisely $\{v_i, v_{i+1}\}$ (with indices taken modulo 5). $B = \cup_i B_i$.
- $U_i$ is the set of vertices in $B_i$ with a neighbour in $N$. $U = \cup_i U_i$.
- For $i = 1, \ldots, 5$, $C_i$ is the set of vertices whose neighbourhood in $H$ is precisely $\{v_i, v_{i+1}, v_{i+2}\}$. Denote $C_i \cup \{v_{i+1}\}$ by $C'_i$ and define $C = \cup_i C_i$ and $C' = \cup_i C'_i = C \cup H$.
- For $i = 1, \ldots, 5$, $D_i$ is the set of vertices whose neighbourhood in $H$ is precisely $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$. $D = \cup_i D_i$.

Since $G$ is claw-free, if a vertex outside $H$ sees $v_i$, then it also sees $v_{i-1}$ or $v_{i+1}$. Thus we can see that the sets $A, B, C', D, N$ partition $V(G)$.

We can begin with some simple observations. Again and throughout this chapter, indices are modulo 5.

- For $i = 1, \ldots, 5$, $B_i \cup C'_i \cup D_i$ is a clique. This follows from the fact that $G$ is claw-free, every vertex in $B_i \cup C'_i \cup D_i$ sees $v_i$, and none sees $v_{i-1}$.

- There is no edge between $B$ and $A$, since a vertex in $B_i$ is nonadjacent to $v_{i-1}$ and $v_{i+2}$, which are nonadjacent to each other but both adjacent to every vertex in $A$. Thus an edge between $B$ and $A$ would imply a claw in $G$.

  By the same reasoning, for $i = 1, \ldots, 5$ there is no edge between $B_i$ and $D_{i-1} \cup D_{i+1} \cup D_{i+2} \cup C'_{i+2}$.

- For $i = 1, \ldots, 5$, $B_i \cup B_{i+1}$ is a clique, since $v_{i+1}$ sees $B_i \cup B_{i+1}$ and $v$.

These facts are symmetric in the sense that just as $B_i \cup C'_i \cup D_i$ is a clique, so is $B_i \cup C'_{i-1} \cup D_{i-2}$.

### 8.2.2 A dominating $W_5$

If $G$ is connected and $U$ is empty, then $N(H)$ is the entire graph and $H$ is dominating. We now prove that in this situation, $\alpha(G) \leq 3$. This was proved by Lovász and Plummer, appearing as Claim 5 in the proof of 12.4.3 in [LP86].

**Theorem 8.9.** *If a claw-free graph $G$ contains a dominating $W_5$ then $\alpha(G) \leq 3$.*

*Proof.* Let $H$ be a dominating 5-hole in the neighbourhood of some vertex $v$. We partition the vertices of $G$ as described above. In this case $N$ is empty and $A$ is nonempty. Suppose $G$ contains a stable set $X$ of size four.

If $|H \cap X| = 2$, then the other two vertices of $X$ are in the clique $B_i$ for some $i$, a contradiction. Suppose then that $H \cap X = v_1$. Then $X - H$ is contained in $B \cup C' \cup D$. More specifically, it is contained in $B_2 \cup B_3 \cup B_4 \cup C'_2 \cup C'_3 \cup D_2$. Since $G$ is claw-free, $v_2$ and $v_5$ each have at most one neighbour in $X - v_1$ while $v_3$ and $v_4$ each have at most two neighbours in $X - v_1$. Since each vertex of $X - v_1$ has at least two neighbours in $H$, it follows that $X - v_1$ consists of one vertex from each of $B_2$, $B_3$, and $B_4$. This contradicts the fact that $B_2 \cup B_3$ and $B_3 \cup B_4$ are cliques. Therefore $X$ and $H$ are disjoint.

There cannot be more then ten edges between $H$ and $X$, otherwise some vertex of $H$ has three neighbours in $X$, giving a claw. Since $\alpha(G[B]) \leq 2$, it follows that $X \subseteq B \cup C'$, and $|X \cap B| = 2$. Assume by symmetry that $X$ contains a vertex of $B_1$ and a vertex of $B_3$. But $B_i$ is complete to both $C'_i$ and $C'_{i-1}$, so $X$ contains two vertices in $C'_4$, a clique. Therefore $X$ cannot exist and $\alpha(G) \leq 3$. □

### 8.2.3 The second neighbourhood of a $W_5$

Every vertex with a neighbour in $H$ and a neighbour in $N$ must see a clique in $H$. Therefore the only vertices with a neighbour in $H$ and a neighbour in $N$ are those vertices in $U$.

Suppose a vertex $w \in N$ has a neighbour in $U_i$. Observe that if $B_{i+1}$ is nonempty, then $w$ must see all of $U_i \cup B_{i+1}$, otherwise there would be a claw. Furthermore the set of vertices of $N$ with a neighbour in $U_i$ form a clique. This simple fact suggests something more significant: By characterizing $U$ based on which sets $U_i$ are empty and nonempty, we can determine how $N(H)$ is attached to the rest of the graph via the second neighbourhood of $H$. Specifically, we deal with two separate cases, depending on whether or not the nonempty $U_i$ sets appear in contiguous order around the cycle. Specifically:

**Definition 8.10.** *The set $U$ is* contiguous *if there is no $i$ such that $U_i$ and $U_{i+2}$ are nonempty but $U_{i-1}$ and $U_{i+1}$ are empty.*

Because two consecutive nonempty $U_i$ and $U_{i+1}$ imply homogeneity of $U_i$ and $U_{i+1}$ outside the neighbourhood of the $W_5$, the way in which $H$ and its neighbourhood are connected to the rest of the graph is strongly influenced by whether or not $U$ is contiguous. In the remainder of this section we will show:

- If $U$ is contiguous, then either $\alpha(G) \leq 3$ or $N(H)$ is separated from the rest of the graph by a 1-join.
- If $U$ is not contiguous, then $N(H)$ is separated from the rest of the graph by one of two specific types of 2-join.

As the decomposition that we find depends on the structure of $U$, our first job is to prove that $\alpha(G) \leq 3$ whenever $U$ is empty.

### 8.2.4 Decomposing around a non-dominating $W_5$

Now suppose $U$ is nonempty, which is equivalent to saying that our $W_5$ is non-dominating. We examine the structure around $H$ in two cases, depending on whether or not $U$ is contiguous. When $U$ is contiguous we get a 1-join or $\alpha(G) = 3$. When $U$ is not contiguous we get one of two types of 2-join.

#### Case 1: $U$ is contiguous

Note that $U$ is a cutset, so if $U$ is a clique it is obviously a clique cutset. Proving that it is one side of a 1-join takes a little more work. We will additionally show that the 1-join has a specific structure with respect to $H$.

**Definition 8.11.** *A* 1*-join $(X, Y)$ is a $W_5$* 1-join *if $X = U$, and $N(H)$ is equal to either $G_2$ or $G_2 \cup U$.*

**Lemma 8.12.** *If $U$ is a clique, then $G$ admits a $W_5$ 1-join $(U, X)$ for some set $X$.*

*Proof.* Assume $U$ is a clique. First suppose $U = U_i$ for some $i$; assume $i = 1$. Since every vertex in $U_1$ has a neighbour in $N$, we can see that $U_1$ is complete to $B_5 \cup B_1 \cup B_2$, $C'_1 \cup C'_5$, and $D_1 \cup D_4$. Likewise $U_1$ is anticomplete to $A$, $D_2 \cup D_3 \cup D_5$, $C'_2 \cup C'_3 \cup C'_4$, and $B_3 \cup B_4$. Letting $X$ be the set of vertices in $V(G) \setminus (N \cup U)$ that are complete to $U$, it is easy to see that $X$ is a clique and so $(U, X)$ is a 1-join.

Now suppose $U$ has vertices $x \in U_i$ and $y \in U_j$ for some nonequal $i$ and $j$. Since any vertex in $U_i$ has neighbourhood $\{v_i, v_{i+1}\}$ in $H$, $x$ does not see every neighbour of $y$ in $H$, and vice-versa. Thus $x$ and $y$ have the same neighbourhood in $N$ since $x$ sees $y$ and $G$ is claw-free. It follows that every vertex in $U$ has the same neighbourhood $X$ in $N$, so $U$ is complete to $X$. Since $G$ is claw-free $X$ is clearly a clique, thus $(U, X)$ is a 1-join. □

We now prove, in several steps, that if $U$ is contiguous but not a clique, then $\alpha(G) = 3$.

**Lemma 8.13.** *Suppose $U$ is contiguous but is not a clique. Then $U = B$ and $N$ is a homogeneous clique.*

*Proof.* Since $U$ is contiguous, we can assume by symmetry that there is a $j \leq 5$ such that $U_i = \emptyset$ precisely if $i > j$. Since $U$ is not a clique, $j \geq 3$.

Suppose that there is a vertex $x$ in $B_i$ for $i \leq j-1$ and a vertex $y \in U_{i+1}$ such that $N(x) \cap N \subset N(y) \cap N$. Then there is a vertex $z \in N$ that sees $y$ but not $x$, so $x$, $y$, and $z$ form a claw with $v_{i+1}$, a contradiction. Therefore for $1 \leq i < j$, every vertex in $N$ seen by a vertex in $U_{i+1}$ is seen by every vertex in $B_i$. By symmetry, for $1 < i \leq j$ every vertex in $N$ seen by a vertex in $U_i$ is seen by every vertex in $B_{i-1}$. It follows that $U = B$ and every vertex in $U$ has the same neighbourhood in $N$. Since $G$ is claw-free, $N \cap N(U)$ is a clique. And since $U$ is not a clique, $N \subseteq N(U)$. □

With this result in hand, we can prove that if $U_i$ is nonempty for all $1 \leq i \leq 5$, then $\alpha(G) \leq 3$ when $U$ is not a clique.

**Lemma 8.14.** *Suppose $U_i$ is nonempty for $i = 1, \ldots, 5$. If $U$ is not a clique, then $\alpha(G) \leq 3$.*

*Proof.* Assume that $U$ is not a clique and there is a stable set $X$ of size four. Theorem 8.9 tells us that $\alpha(G - N) \leq 3$, so $X$ must contain a vertex of $N$. Thus $X$ does not intersect $B$ since by the previous lemma $B = U$.

If $X$ contains two vertices of $H$, then every vertex in $C' \cup D \cup A$ sees at least one of these vertices, so $X$ cannot be a stable set of size four. Suppose then that $X \cap H = \{v_1\}$. There are two vertices of $X$ in $C' \cup D \cup A$, and they must both be in $C'_2 \cup C'_3 \cup D_2$. We already know that $C'_2 \cup D_2$ and $C'_3 \cup D_2$ are cliques, so $X$ must contain two vertices of $C'_2 \cup C'_3$. But $B_2$ is nonempty and sees $C'_2 \cup C'_3 \cup N$, contradicting the fact that $G$ is claw-free. Thus $X \cap H = \emptyset$.

Since $B_i$ is nonempty for every $i$, we can see that $C'_i \cup C'_{i+1}$ is a clique for every $i$, otherwise there would be a claw containing a vertex of $B_i$. It follows that $\alpha(C') \leq 2$, so $|X \cap C'| \leq 2$. And since there are at most ten edges between $X \cap (C' \cup D \cup A)$ and $H$, we see that $X \cap A = \emptyset$ and $|X \cap C'| \geq 2$, so $|X \cap C'| = 2$ and $|X \cap D| = 1$. Thus we can assume that $X$ contains a vertex of $C'_1$ and a vertex of $C'_3$, and since $G$ is claw-free the vertex in $X \cap D$ must be in $D_4$. But every vertex of $B_1$ is complete to $C'_1 \cup D_4 \cup U$, giving us a claw and a contradiction. Therefore $X$ cannot exist. □

**Lemma 8.15.** *If $U$ is contiguous and is not a clique, then $\alpha(G) \leq 3$.*

*Proof.* By the previous lemma, assume that $U$ is not a clique, $j \in \{3, 4\}$, and $U_i$ is nonempty precisely when $i \leq j$. We claim that in this case $G$ has a dominating $W_5$, which implies the result by Theorem 8.9.

Take $v_0 \in A$, $u_2 \in U_2$, and $u_3 \in U_3$. The vertices $v_2u_2u_3v_4v_0$ induce a $C_5$ in the neighbourhood of $v_3$. It is simple to confirm that this 5-wheel is dominating if $B_5$ is empty. Lemma 8.13 implies $B_5 = U_5 = \emptyset$, so we are done. □

We have just proved that $\alpha(G) = 3$ if $G$ has a nondominating 5-wheel for which $U_i$ is nonempty precisely when $i \leq j$ for some $j \in \{3, 4, 5\}$. In these cases $G$ contains, as an induced subgraph, a large (i.e. at least ten vertices) induced subgraph of the icosahedron. It is possible to prove that in this situation $G$ is very close to being a thickening of this induced subgraph of the icosahedron. More specifically, $G$ is an *icosahedral thickening*; we define these graphs in the next chapter. We omit the straightforward details of this characterization, which was done by Chudnovsky and Seymour in Section 5 of [CS07c].

**Case 2: $U$ is not contiguous**

If $U$ is not contiguous there are two cases to consider, by symmetry. In the first case, $U_i \neq \emptyset$ if and only if $i \in \{1, 3\}$. In the second, $U_i \neq \emptyset$ if and only if $i \in \{1, 3, 4\}$. We will prove that these two cases result in two types of 2-join, which we define now.

**Definition 8.16.** *A 2-join $((X_1, Y_1), (X_2, Y_2))$ is a* $W_5$ 2-join *if there is an induced 5-hole $H$ in the neighbourhood of some vertex $v$ such that $G_2 \cup X_1 \cup Y_1 = N(H)$ and with respect to $H$, $U = U_1 \cup U_3$ where $U_1$ and $U_3$ are nonempty.*

**Definition 8.17.** *A 2-join $((X_1, Y_1), (X_2, Y_2))$ is a* gear 2-join *if there is an induced 5-hole $H$ in the neighbourhood of some vertex $v$ such that $G_2 \cup Y_1 = N(H)$ and with respect to $H$, $U = U_1 \cup U_3 \cup U_4$ where $U_1$, $U_3$, and $U_4$ are nonempty.*

Galluccio, Gentile, and Ventura [GGV08] investigated gear 2-joins and $W_5$ 2-joins (and introduced the term *gear*), describing facets of the stable set polytope of claw-free graphs with $\alpha(G) \geq 4$.

Denote by $N_1$ those vertices of $N$ with a neighbour in $U_1$, and denote by $N_3$ those vertices of $N$ with a neighbour in $U_3 \cup U_4$. Denote by $U_1'$ those vertices of $B_1$ with a neighbour in $B_3 \cup B_4$.

**Lemma 8.18.** *Suppose $U_i$ is nonempty precisely for $i \in \{1, 3, 4\}$. Then $G$ admits a gear 2-join.*

*Proof.* We claim that

$$(\ (U_1 \cup U_1',\ N_3 \cup U_1'),\ ((B_1 \setminus (U_1 \cup U_1')) \cup C_5' \cup C_1',\ U_3 \cup U_4)\ )$$

is a 2-join; it is easy to confirm that it is a gear 2-join. Since $B_3$ is nonempty, $D_1$ must be complete to $A$, but since $B_1$ and $B_3$ are nonempty there is a claw at $D_1$ if $D_1$ is nonempty. Thus by symmetry $D_1$ and $D_4$ are both empty. We already know that $B_1$ is complete to $C_5' \cup C_1'$. If $v \in U_1'$ then $v$ is complete to $N_3 \cup B_3 \cup B_4$ otherwise there would be a claw at $B_3 \cup B_4$. Furthermore $N_3$ is complete to $U_3 \cup U_4$. Since no vertex outside $B$ sees a vertex in $N$, we have the desired gear 2-join. □

**Lemma 8.19.** *Suppose $U_i$ is nonempty precisely for $i \in \{1, 3\}$. Then $G$ admits a $W_5$ 2-join.*

*Proof.* Now we assume $U_4$ is empty. In this case, $U_1$ is complete to $B_1'$, $C_1'$, $C_5'$, $D_1$, and $D_4$, and anticomplete to $A$, $C_2'$, $C_3'$, $C_4'$, $D_2$, $D_3$, and $D_5$. Similarly, $U_3$ is complete to $B_3'$, $C_3'$, $C_2'$, $D_3$, and $D_1$, and anticomplete to $A$, $C_4'$, $C_5'$, $C_1'$, $D_4$, $D_5$, and $D_2$. Thus again we have a 2-join:

$$(\ (U_1,\ U_3),\ (B_1' \cup C_1' \cup C_5' \cup D_1 \cup D_4,\ B_3' \cup C_2' \cup C_3' \cup D_3 \cup D_1)\ ).$$

It is easy to confirm that it is a $W_5$ 2-join. □

Combining the results from this chapter gives us a decomposition theorem, which corresponds to Lemma 3.6 in [OPS08].

**Theorem 8.20.** *Let $G$ be a connected claw-free graph with $\alpha(G) \geq 4$. If $G$ is not quasi-line then $G$ admits a $W_5$ 1-join or a $W_5$ 2-join or a gear 2-join.*

*Proof of Theorem 8.20.* Suppose $G$ is claw-free but not quasi-line, and suppose $\alpha(G) \geq 4$. If $U$ is contiguous we know it must be a clique, but then $G$ admits a $W_5$ 1-join. If $U$ is not contiguous then it admits either a $W_5$ 2-join or a gear 2-join. □

We have just shown that if $\alpha(G) \geq 4$, the 5-wheels of $G$ lie in clusters that can be separated from the rest of the graph by 1-joins and 2-joins. In the next section we will turn this decomposition theorem into a structure theorem.

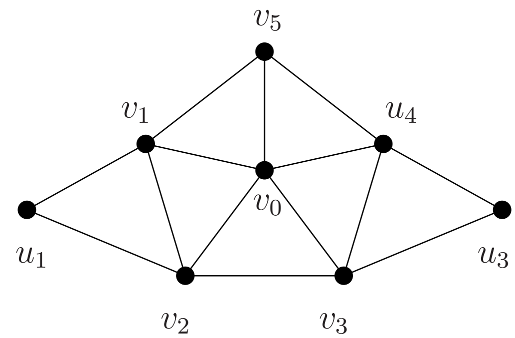


Figure 8.3: The graph $H_1$ from which $W_5$ strips are built.

## 8.3 Reducing on a $W_5$ 2-join

In this section we extend Theorem 5.15, the structure theorem for quasi-line graphs. We use Theorem 5.15 to prove the following: By allowing "$W_5$ strips" and "gear strips" (which we define shortly) as well as fuzzy linear interval strips, we can construct any claw-free graph with $\alpha(G) \geq 4$ that does not admit a 1-join and is not a fuzzy circular interval graph.

**Theorem 8.21.** *Let $G$ be a claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- *$G$ admits a 1-join*
- *$G$ is a fuzzy circular interval graph*
- *$G$ is a composition of fuzzy linear interval strips, $W_5$ strips, and gear strips.*

First we must define $W_5$ strips and gear strips, which are the simplest strips that result in $W_5$ 2-joins and gear 2-joins under the composition operation.

Let $H_1$ be the graph on vertices $v_0, v_1, \ldots v_5, u_1, u_3$ such that $v_1, \ldots, v_5$ induce, in order, a $C_5$ in the neighbourhood of $v_0$. Let $u_1$ and $u_3$ have neighbourhoods $\{v_1, v_2\}$ and $\{v_3, v_4\}$ respectively (see Figure 8.3). Now suppose $G$ is a claw-free graph containing $H_1$ as an induced subgraph, such that $u_1$ and $u_3$ are simplicial in $G$ and every vertex in $G$ has a neighbour in $\{v_1, \ldots, v_5\}$. Then $(G - \{u_1, u_3\}, N(u_1), N(u_3))$ is a strip and we say it is a $W_5$ *strip*.

Construct $H_2$ from $H_1$ by adding vertices $u_3$ and $x$ with neighbourhoods $\{v_4, v_5, u_3\}$ and $\{u_3, u_4\}$ respectively (see Figure 8.4). Suppose $G$ is a claw-free graph containing $H_2$ as an induced subgraph, such that $u_1$ and $x$ are simplicial in $G$ and every vertex in $G$ has a neighbour in $\{v_1, \ldots, v_5, u_3, u_4\}$. Then $(G - \{u_1, x\}, N(u_1), N(x))$ is a strip and we say it is a *gear strip*.

We are now ready to prove the structure theorem.

*Proof of Theorem 8.21.* Let $G$ be a minimum counterexample to the theorem. We know that $G$ is not a quasi-line graph, therefore it contains a $W_5$ which we call $H$. By the results of the previous section, we know that $H$ is nondominating and that $G$ admits either a $W_5$ 2-join or a gear 2-join, $((X_1, Y_1), (X_2, Y_2))$, separating $G_1$ from $G_2$ and such that $H$ is contained in $G_2$. We will prove that $G$ is a composition of strips in which the vertices of $G_2$ arise from a single $W_2$ strip or gear strip.

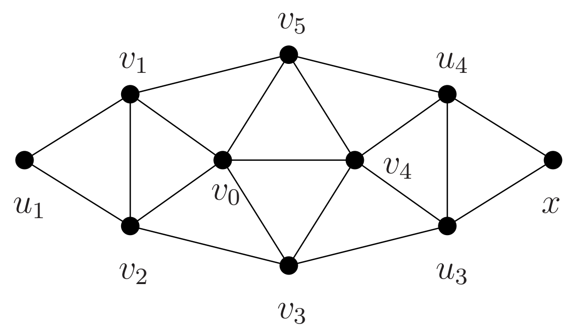


Figure 8.4: The graph $H_2$ from which gear strips are built.

To do this we first construct a graph $G'$ from $G_1$ by adding a path $G'_2$ on five vertices $p_1, \dots, p_5$ in order such that $p_1$ sees all of $X_1$, $p_5$ sees all of $Y_1$, and there are no other edges between $G_1 = G'_1$ and $G'_2$. Observe first that $\alpha(G') \geq 4$ since $H$ is nondominating in $G$, and second that $|V(G')| < |V(G)|$. Finally, observe that since $G$ admits no 1-join, $G'$ admits no 1-join: a 1-join in $G'$ would have to be contained in $G'_2$, but this is clearly impossible.

Thus by the minimality of $G$, $G'$ is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips, $W_5$ strips, and gear strips.

Suppose first that $G'$ is a fuzzy circular interval graph. We make two observations. First, $G'$ contains a hole of length at least five, so it cannot be a fuzzy linear interval graph. Second, no vertex in $G'_2$ is in a homogeneous pair of cliques, so since $G'_2$ is a path, the vertices $p_1, \dots, p_5$ are in consecutive circular order in any circular interval representation of $G'$. It follows that $G_1$ is a fuzzy linear interval graph, and it has a linear interval representation with $X_1$ at the far left and $Y_1$ at the far right. Therefore $G'$ is a composition of two fuzzy linear interval strips: $(G_1, X_1, Y_1)$ and $(G'_2, \{p_1\}, \{p_5\})$. Thus $G$ is a composition of two strips: $(G_1, X_1, Y_1)$ and $(G_2, X_2, Y_2)$.

We can therefore assume that $G'$ is not a fuzzy circular interval graph, so it is a composition of fuzzy linear interval strips, $W_5$ strips, and gear strips. It is straightforward to confirm that no vertex in a $W_5$ strip or a gear strip has a disconnected neighbourhood, so no vertex of $G'_2$ is in a $W_5$ strip or a gear strip.

Now we claim that there is a strip representation for $G'$ in which one of the strips is $(G'_2, \{p_1\}, \{p_2\})$. To see this, consider a strip representation for $G'$ in which $p_i$ and $p_{i+1}$ are in different fuzzy linear interval strips $(S_1, \{a\}, \{b\})$ and $(S_2, \{c\}, \{d\})$ respectively. Since $p_i$ and $p_{i+1}$ are adjacent, we can assume that $b = p_i$ and $c = p_{i+1}$. Observe that removing $(S_1, \{a\}, \{b\})$ and $(S_2, \{c\}, \{d\})$ from the strip representation and replacing it with $(S_1 \cup S_2, a, d)$ again yields a strip representation for $G'$. By repeating this argument we can find a strip representation for $G'$ in which every vertex of $G'_2$ is in the same fuzzy linear interval strip.

So consider this strip representation and let $(S, X, Y)$ be the fuzzy linear interval strip from which the vertices of $G'_2$ arise. We will break the strip into three smaller strips to get the desired strip representation of $G'$. There is a fuzzy linear interval representation of $S$ in which $p_1, \dots, p_5$ appear in consecutive linear order; let $S_L$ and $S_R$ be the vertices of $S$ to the left of $p_1$ and the right of $p_5$ respectively. Then it is simple to confirm that replacing $(S, X, Y)$ with the three strips $(S_L, X, X_1)$, $(G'_2, \{p_1\}, \{p_2\})$, $(S_R, Y_1, Y)$ (omitting any empty strips) gives us a strip representation for $G'$.

From this strip representation for $G'$ we now construct a strip representation for $G$. We need only replace $(G'_2, \{p_1\}, \{p_2\})$ with $(G_2, X_2, Y_2)$. Observe that this is equivalent to replacing $G'_2$ with $G_2$ in the graph. Thus $G$ is a composition of fuzzy linear interval strips, $W_5$ strips, and gear strips. This proves the theorem. □

**Remark:** We replace $G_2$ with a path on five vertices to ensure that $\alpha(G') \geq 4$. The natural thing to do is to replace $G_2$ with a path on three vertices, in which case $G'$ is actually an induced subgraph of $G$ whether we have a $W_5$ 2-join or a gear 2-join. This would require more careful consideration of compositions of two strips.

## 8.4 Applying the decomposition

We conclude the chapter with a brief description of two recent applications of the decomposition described in this chapter. The first is an $O(n^6)$ algorithm for finding a maximum weight stable set in a claw-free graph due to Oriolo, Pietropaoli, and Stauffer [OPS08]; the decomposition first appeared in this paper.

Their approach is straightforward, and similar to our approach to fractionally colouring quasi-line graphs. Suppose $G$ is a composition of strips, one of which is $(G_s, X_s, Y_s)$. A stable set in $G$ intersects $G_s$ in $X_s$, or $Y_s$, or both, or neither. For each of these intersection cases, find a maximum weight stable set in $G_s$ subject to the intersection restriction. Call the weights $w_{X,Y}$, $w_{\bar{X},Y}$, $w_{X,\bar{Y}}$, and $w_{\bar{X},\bar{Y}}$, labelled according to whether or not the stable set intersects $X_s$ or $Y_s$ (e.g. the heaviest stable set in $G_s$ intersecting both $X_s$ and $Y_s$ has weight $w_{X,Y}$).

Having computed these values for $G_s$, they construct $G'$ from $G$ by replacing $G_s$ with a three vertex strip $(G'_s, \{x, z\}, \{y, z\})$, weighting the vertices $x$, $y$, and $z$ as $w_{X,\bar{Y}} - w_{\bar{X},\bar{Y}}$, $w_{X,Y} - w_{\bar{X},\bar{Y}}$, and $w_{\bar{X},Y} - w_{\bar{X},\bar{Y}}$, respectively. It is not hard to see that $\alpha_w(G')$, the maximum weight of a stable set in $G'$, is exactly $\alpha_w(G) - w_{\bar{X},\bar{Y}}$.

It is easy to solve the maximum weight stable set problem for graphs with stability number at most three in $O(n^3)$ time. Thus this approach of strip replacement can be used to reduce the maximum weight stable set problem on claw-free graphs to that on quasi-line graphs.

To solve the problem on quasi-line graphs first note the following. Given a nonlinear homogeneous pair of cliques in a quasi-line graph, we can remove an edge from the graph without changing the fact that it is quasi-line or without changing the maximum weight of a stable set. Thus by the structure theorem for quasi-line graphs, only circular interval graphs and compositions of linear interval strips remain. Second, circular interval graphs belong in a class called "distance claw-free", for which Pulleyblank and Shepherd provided and $O(n^3)$ algorithm for the maximum weighted stable set problem [PS93]. So we are left to deal with compositions of linear interval strips. Given a strip decomposition, we can reduce each strip, thereby reducing the problem to finding a maximum weight stable set in a line graph. This problem is equivalent to finding a maximum weight matching in a graph, so we can apply Edmonds' matching algorithm [Edm65b] to complete the solution.

In [GGV08], Galluccio, Gentile, and Ventura investigated polyhedral properties of claw-free graphs, in particular the effect of gear strips. The stable set polytope of quasi-line graphs was recently described in two cases by Chudnovsky and Seymour [CS05] and Eisenbrand, Oriolo, Stauffer, and Ventura [EOSV05]. Galluccio, Gentile, and Ventura conjectured that the stable set polytope of a claw-free graph $G$ that has $\alpha \geq 4$ and is not quasi-line can be described by nonnegativity inequalities, rank inequalities, and two separate types of inequalities that they call *lifted 5-wheel inequalities* and *lifted geared inequalities.* They arise in compositions of strips containing $W_5$ strips and gear strips, respectively, and it was proved that lifted $W_5$ inequalities alone are not sufficient [GGV08].

# Chapter 9

# The structure of claw-free graphs

In this chapter we refine the results of the previous chapter, giving a detailed characterization of skeletal claw-free graphs. We do this by adapting Chudnovsky and Seymour's more general characterization of claw-free trigraphs [CS08b].

We have already shown that compositions of $W_5$ strips, gear strips, and linear interval strips provide a natural way to decompose claw-free graphs when $\alpha \geq 4$. However, our description of $W_5$ strips and gear strips is not very detailed, and in particular it is not detailed enough to let us prove our colouring results in the next chapter. Therefore we will discuss Chudnovsky and Seymour's more thorough investigation of the structure of these strips, and that will allow us to bound the chromatic number as desired.

There is one other composition that we will use to build claw-free graphs with a three-colourable complement. These compositions are called *worn hex-joins*, and require several base classes of claw-free graphs that we will define in Section 9.3.2.

We begin by describing some fundamental types of claw-free graphs. These let us characterize $W_5$ strips, gear strips, and the base classes of worn hex-joins, but they also cover skeletal claw-free graphs that do not arise as a composition of strips or a worn hex-join.

In the next chapter we will use these refined structural results to bound the chromatic number of claw-free graphs.

## 9.1 Some important types of claw-free graphs

Here we present several fundamental classes of claw-free graphs. We will use them to describe the base classes for our two composition operations.

### 9.1.1 Long circular interval graphs

We discussed circular interval graphs in Chapter 4. A circular interval graph is a *long circular interval graph* if it has a circular interval representation in which no three intervals cover the entire circle. Note that it is still possible for three intervals to cover all vertices. Just as a fuzzy circular interval graph is a thickening of a circular interval graph, a *long fuzzy circular interval graph* is a thickening of a long circular interval graph. It is not hard to show that if $G$ is a thickening of a long circular interval graph $G'$ under a claw-neutral matching $M$, then we can assume that each edge of $M$ corresponds to the two extreme vertices of an interval. Using this fact, one can easily see that a skeletal long fuzzy circular interval graph is a long circular interval graph. Chudnovsky and Seymour described the structure of these graphs in depth and provided a forbidden subgraph characterization [CS08a].

### 9.1.2 Nearly-line thickenings

We now define a slight generalization of line graphs. They are *nearly-line thickenings*, which are special thickenings of line graphs.

Suppose $G = L(H)$ for some graph $H$. Let $M$ be a matching of $G$ such that for any $uv \in M$, the edges of $H$ corresponding to $u$ and $v$ share an endpoint of degree two in $H$. In this case $M$ is a claw-neutral matching, so any thickening $G'$ of $G$ under $M$ is claw-free. We call such a graph $G'$ a *nearly-line thickening*.

If $M$ is empty then $G'$ is a line graph, and if $M$ is nonempty then $G'$ admits a linear interval 2-join.

**Lemma 9.1.** *Every skeletal nearly-line thickening is a line graph.*

This result is based on an important property of skeletal homogeneous cliques. For a thickening of $H$ under a matching $M$, if $v_1v_2 \in M$ and $(I(v_1), I(v_2))$ is skeletal, then we can simply replace $v_1$ and $v_2$ in $H$ by four vertices corresponding to $I(v_1) \cap \Omega(v_1v_2)$, $I(v_1) \setminus \Omega(v_1v_2)$, $I(v_2) \cap \Omega(v_1v_2)$, and $I(v_1) \setminus \Omega(v_1v_2)$. The resulting graph will be a line graph, just as $H$ is.

*Proof.* Suppose $G$, a skeletal nearly-line thickening, is a thickening of a line graph $H$ under a claw-neutral matching $M$. We choose $H$ to be minimal, so we can assume that $H$ is the line graph of a simple graph $J$. We proceed by induction on $|M|$. If $M = \emptyset$ then $G$ is clearly a line graph.

Let $e_1, e_2$ be edges of $J$ corresponding to vertices $v_1, v_2$ of $H$ with $v_1v_2$ in $M$. We construct the graph $J'$ from $J$ as follows. Make two new pendant edges $e'_1, e'_2$ of $J$, such that $e'_1$ (resp. $e'_2$) is incident to the endpoint of $e_1$ (resp. $e_2$) not shared by $e_2$ (resp. $e_1$). If $I(v_1) \subseteq \Omega(v_1v_2)$ then delete $e'_1$, and if $I(v_2) \subseteq \Omega(v_1v_2)$ then delete $e'_2$. Denote $L(J')$ by $H'$ and call the vertices corresponding to $e'_1$ and $e'_2$, $v'_1$ and $v'_2$ respectively.

We claim that $G$ is a thickening of $H'$ under $M - \{v_1v_2\}$, which implies the lemma by induction on $|M|$. To see this observe that the thickening will remain the same except on $I(v_1)$, $I(v'_1)$, $I(v_2)$, and $I(v'_2)$. Simply let $I(v'_1) = I(v_1) \setminus \Omega(v_1v_2)$ and let $I(v'_2) = I(v_2) \setminus \Omega(v_1v_2)$. □

### 9.1.3 Antiprismatic thickenings

A *triad* is a stable set of size three. A graph $G$ is *antiprismatic* if every triad $T$ contains exactly two neighbours of every vertex in $G - T$. Such graphs are clearly claw-free. Let $M$ be a matching in $G$ such that $G - M$ is also antiprismatic. Then $M$ is claw-neutral in $G$. If $G'$ is a thickening of an antiprismatic graph $G$ under such a matching $M$, then we say that $G'$ is an *antiprismatic thickening*.

Just like skeletal nearly-line thickenings, skeletal antiprismatic thickenings behave nicely:

**Lemma 9.2.** *Every skeletal antiprismatic thickening is a proper thickening of an antiprismatic graph.*

The proof of this result uses the same idea as the proof for nearly-line thickenings.

*Proof.* Suppose $G$, a skeletal antiprismatic thickening, is a thickening of an antiprismatic graph $H$ under a matching $M$. Again we proceed by induction on $|M|$, so assume $M$ is minimum. Clearly if $M = \emptyset$ then $G$ is a proper thickening of $H$ and we are done.

Take vertices $v_1$ and $v_2$ of $H$ such that $v_1v_2 \in M$. Then $(I(v_1), I(v_2))$ is a skeletal homogeneous pair of cliques. Construct $H'$ from $H$ by adding vertices $v_1'$ and $v_2'$ with neighbourhoods $\bar{N}(v_1)\setminus\{v_2\}$ and $\bar{N}(v_2) \setminus \{v_1\}$ respectively. We claim that $G$ is a thickening of $H'$ under $M - \{v_1v_2\}$ and that $H'$ is antiprismatic.

To see that $H'$ is antiprismatic, we recall that by the definition of an antiprismatic thickening, $H - v_1v_2$ is antiprismatic. Chudnovksy and Seymour characterized "changeable edges" of this type (§16 in [CS07a]), proving that in this case neither $v_1$ nor $v_2$ is in a triad. It easily follows that $H'$ is antiprismatic.

As in the previous proof, to see that $G$ is a thickening of $H'$ under $M - \{v_1v_2\}$ we use the same thickening, except we set $I(v_1') = I(v_1) \setminus \Omega(v_1v_2)$ and $I(v_2') = I(v_2) \setminus \Omega(v_1v_2)$. □

This proof, along with Lemma 6.12, provides a useful corollary:

**Corollary 9.3.** *Suppose $G$ is an antiprismatic thickening of an antihat $H$ under a matching $M$ of $H$. Then there is a subgraph $G'$ of $G$ such that $G'$ is a proper thickening of an antihat graph $H'$, and $\chi(G) = \chi(G')$.*

Antiprismatic thickenings are very easy to define, but describing their structure is a different matter altogether. Chudnovsky and Seymour provided a lengthy and difficult characterization of their structure [CS07a, CS07b], but we do not need to use it in this thesis. In fact our desire to avoid it inspired us to consider both skeletal graphs and the Local Strengthening. Proving the Local Strengthening for these graphs is very easy given Lemma 9.2.

It is important to bear in mind that any graph $G$ with $\alpha(G) \leq 2$ is trivially antiprismatic, just as it is trivially claw-free.

### 9.1.4 Antihat thickenings

We need to consider certain thickenings of graphs that are nearly antiprismatic. Let $k \geq 2$. We first define a graph $H$ with vertex set $A \cup B \cup C$ as follows. Let $A = \{a_0, a_1, \dots, a_k\}$, $B = \{b_0, b_1, \dots, b_k\}$, and $C = \{c_1, \dots, c_k\}$ be disjoint cliques. Adjacency between the cliques is as follows:

- $a_0$ has no neighbour outside $A \cup \{b_0\}$, and $b_0$ has no neighbour outside $B \cup \{a_0\}$.
- For $1 \leq i, j \leq k$, $a_i$ and $b_j$ are nonadjacent if $i \neq j$ and adjacent if $i = j$.
- For $1 \leq i, j \leq k$, $a_i$ and $b_i$ are adjacent to $c_j$ if $i \neq j$, and nonadjacent to $c_j$ if $i = 0$ or if $i = j$.

Let $X \subset A \cup B \cup C \setminus \{a_0, b_0\}$ such that $|C \setminus X| \geq 2$, and let $G = H - X$. We say that $G$ is an *antihat graph.* To define antihat thickenings, we first define a set $M \in V(G)^2$ as follows:

- $M$ is a matching in $G \cup M$ containing no edge of $G[A]$, $G[B]$, or $G[C]$.
- $a_0b_0$ is in $M$ if $a_0$ and $b_0$ are adjacent in $G$.
- If $1 \leq i, j$ and $a_ib_j \in M$ then $i = j$ and $c_i \in X$.

- If $1 \leq i, j$ and $b_i c_j \in M$ then $i = j$ and $a_i \in X$.
- If $1 \leq i, j$ and $a_i c_j \in M$ then $i = j$ and $b_i \in X$.

In this case $G \cup M$ is claw-free and $M$ is a claw-neutral matching in $G \cup M$. If $G'$ is a thickening of $G \cup M$ under $M$ then we say that it is an *antihat thickening*.

Observe that given an antihat graph $G$, adding an edge between $a_0$ and $b_0$ gives us an antiprismatic graph, as does deleting one or both of $a_0$ and $b_0$.

### 9.1.5 Icosahedral thickenings

The icosahedron is the unique vertex-transitive graph on twelve vertices in which the neighbourhood of every vertex induces a $C_5$. Theorem 8.1 tells us that a claw-free graph with $\alpha \geq 3$ is quasi-line precisely if no neighbourhood contains an induced $C_5$, so the icosahedron is the epitome of a claw-free graph that is not quasi-line.

There are several graphs related to the icosahedron that we must treat as a structural exception. These were hinted at in Section 8.2.4. The first is the icosahedron itself, which we define explicitly. Let the graph $G_0$ have vertices $v_0, v_1, \ldots, v_{11}$. For $i = 1, \ldots, 10$, $v_i$ is adjacent to $v_{i+1}$ and $v_{i+2}$ with indices modulo 10. The neighbourhood of $v_0$ is $\{v_i : 1 \leq i \leq 10,\ i \text{ is odd}\}$, and the neighbourhood of $v_{11}$ is $\{v_i : 1 \leq i \leq 10,\ i \text{ is even}\}$. $G_0$ is the icosahedron (see Figure 9.1).

We obtain $G_1$ from $G_0$ by deleting $v_{11}$, and we obtain $G_2$ from $G_1$ by deleting $v_{10}$. Note that if $M \in V(G_2)^2$ is a subset of $\{v_1v_4, v_6v_9\}$, then $M$ is a claw-neutral matching in $G_2 \cup M$. We say that $G'$ is an *icosahedral thickening* if it is a proper thickening of $G_0$ or $G_1$, or is a thickening of $G_2 \cup M$ under some $M \subseteq \{v_1v_4, v_6v_9\}$. Any icosahedral thickening $G'$ has $\alpha(G') = 3$ and $\chi(\overline{G'}) = 4$.

## 9.2 Claw-free graphs with $\alpha \geq 4$

In the previous chapter we proved that any claw-free graph containing a stable set of size four is quasi-line or admits a 1-join or is a composition of fuzzy linear interval strips, gear strips, and $W_5$ strips. To prove our colouring results we need a better description of $W_5$ strips and gear strips. We now present such a description that follows from Chudnovsky and Seymour's work [CS08b].

### 9.2.1 Five types of strips

In total we need to consider five types of strips. The first is linear interval strips, which we already defined and used when considering quasi-line graphs. Now we define the other four. Their structure will be enough to let us prove the Main Conjecture for compositions of strips.

#### Antihat strips

Let $G$ be an antihat graph, and let $G'$ be an antihat thickening, i.e. a thickening of $G \cup M$ under $M$ as defined in Section 9.1.4. We specify two cliques of $G'$: $A' = I(A \setminus (X \cup \{a_0\}))$ and $B' = I(B \setminus (X \cup \{b_0\}))$. Then $(G' - I(a_0) - I(b_0), A', B')$ is a strip and if it contains a $W_5$ we say that it is an *antihat strip*. These antihat strips are a slight generalization of the antihat strips used in Chudnovsky and Seymour's survey [CS05].

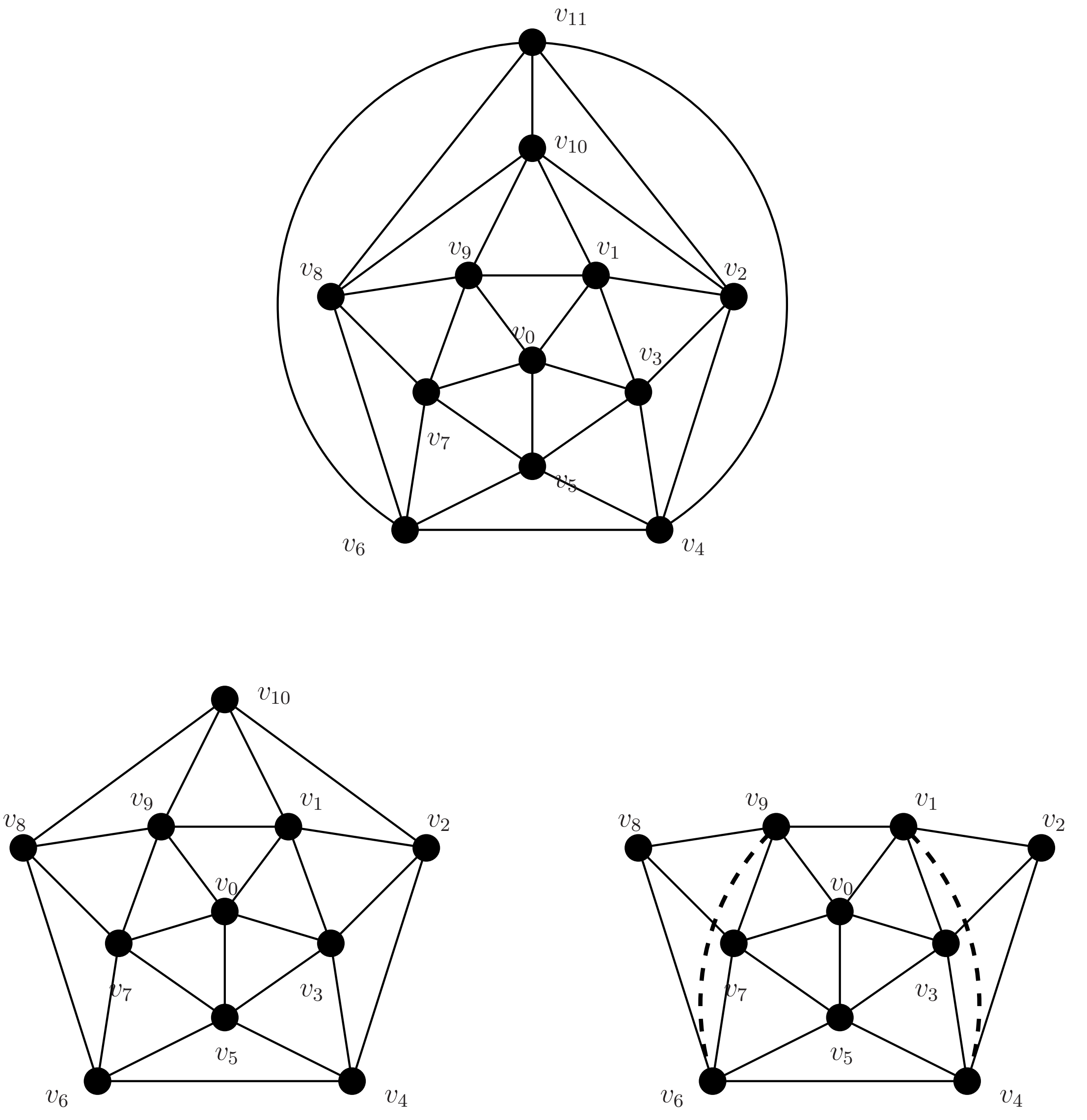


Figure 9.1: The icosahedron $G_0$ (top), with its derivative graphs $G_1$ (left) and $G_2 \cup M$ (right). In $G_2 \cup M$, each of $\{v_1, v_4\}$ and $\{v_6, v_9\}$ is a nonadjacent pair or is in $M$.

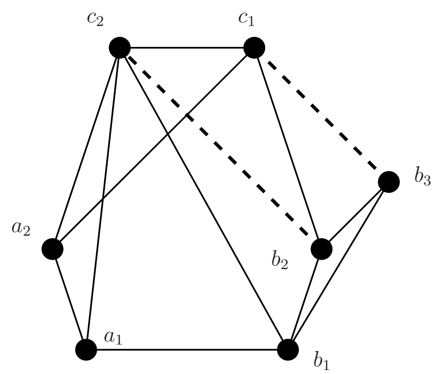


Figure 9.2: The graph underlying strange strips. Dashed lines represent edges in $M$.

### Pseudo-line strips

We will define a type of line graph and modify it slightly. Let $J$ be a graph containing a path on vertices $j_1, j_2, j_3$ in order such that every edge of $J$ is incident to at least one of $j_1, j_2, j_3$. Let $H = L(J)$, and for $i \in \{1, 3\}$ let $X_i$ be the set of vertices of $H$ corresponding to edges incident to $j_i$ in $J$. Both $X_1$ and $X_3$ are cliques. Let $v_1$ and $v_2$ be the vertices of $H$ corresponding to the edges $j_1j_2$ and $j_2j_3$ respectively. Let $G$ be a thickening of $H$ under $M = \{v_1v_2\}$. Then $(G, X_1, X_3)$ is a strip and if it contains a $W_5$ we say it is a *pseudo-line strip*.

These strips correspond to thickenings of the class $\mathcal{Z}_3$ defined in [CS08b]. We call the vertices of $J$ other than $\{j_i \mid 1 \leq i \leq 3\}$ *centres* of $J$.

### Strange strips

Let $H$ be a claw-free graph on cliques $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$ with adjacency as follows: $a_1, b_1$ are adjacent; $c_1$ is adjacent to $a_2$ and $b_2$ and $b_3$; $c_2$ is adjacent to $a_1$, $a_2$, $b_1$, and $b_2$. All other pairs are nonadjacent. Let $G$ be a thickening of $H$ under $M = \{b_3c_1, b_2c_2\}$ (see Figure 9.2). Then $(G, I(A), I(B))$ is a strip and we say that it is a *strange strip*.

### Gear strips

Now we give a full description of the structure of gear strips – there is only one type. We begin with a graph $H$ on vertices $\{v_1, \ldots, v_{10}\}$ with adjacency as follows. The vertices $v_1, \ldots, v_6$ are a 6-hole in order. Next, $v_7$ is adjacent to $v_1, v_2, v_3, v_6$; $v_8$ is adjacent to $v_3, v_4, v_5, v_6, v_7$; $v_9$ is adjacent to $v_3, v_4, v_6, v_1, v_7, v_8$; $v_{10}$ is adjacent to $v_2, v_3, v_5, v_6, v_7, v_8$. There are no other edges in $H$. Let $X \subseteq \{v_9, v_{10}\}$. See Figure 9.3.

If $G$ is a thickening of $H \setminus X$ under a matching $M \subseteq \{v_7v_8\}$, then $(G, I(v_1) \cup I(v_2), I(v_4) \cup I(v_5))$ is a strip, and we say that it is a *gear strip*. These correspond to thickenings of the class $\mathcal{Z}_5$ in [CS08b] and are a slight generalization of thickenings of XX-strips as defined in [CS05]. In the definition of $\mathcal{Z}_5$ there is another vertex with neighbourhood $\{v_1, v_2, v_4, v_5\}$, but we can consider this vertex to be a part of a separate strip.

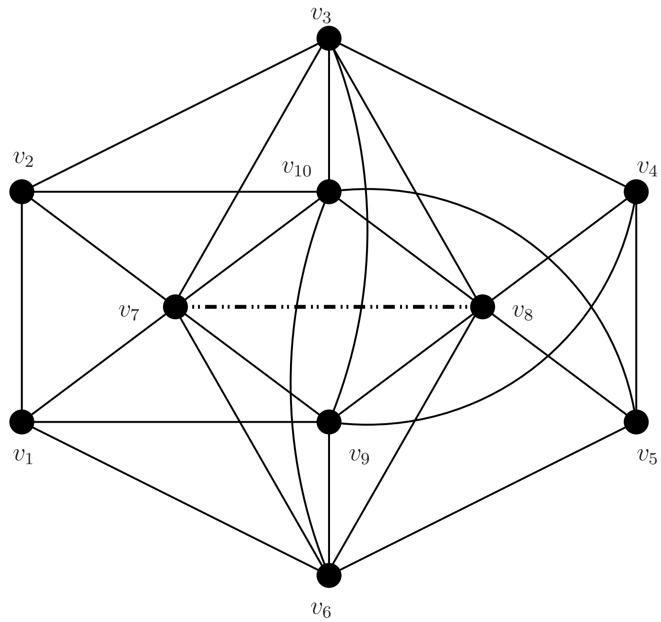


Figure 9.3: The graph underlying gear strips.

### 9.2.2 Five types of 2-joins

Just as we defined interval 2-joins using linear interval strips in Chapter 5, we will define 2-joins corresponding to our other strips. These will allow us to prove the Main Conjecture for these graphs using a minimum counterexample approach, just as we did for quasi-line graphs in Section 7.1.1.

Suppose in our claw-free graph $G$ there is a 2-join $((X_1, Y_1), (X_2, Y_2))$ separating $G_1$ and $G_2$, such that $X_1$, $X_2$, $Y_1$, and $Y_2$ are cliques and are pairwise disjoint except for possibly $X_1$ and $Y_1$.

- **Antihat 2-joins.** If $(G_2, X_2, Y_2)$ is an antihat strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is an *antihat 2-join.*

- **Pseudo-line 2-joins.** If $(G_2, X_2, Y_2)$ is a pseudo-line strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *pseudo-line 2-join.*

- **Strange 2-joins.** If $(G_2, X_2, Y_2)$ is a strange strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *strange 2-join.*

- **Gear 2-joins.** If $(G_2, X_2, Y_2)$ is a gear strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *gear 2-join.*

If $G$ arises as a composition of strips, one of which is a pseudo-line strip $(G_2, X_2, Y_2)$, $G$ will admit a pseudo-line 2-join with $X_2$ and $Y_2$ disjoint – we can simply consider their intersection to be in $X_1 \cap Y_1$ instead, as we did with canonical interval 2-joins.

These types of 2-joins allow us to use a decomposition theorem analogous to the structure theorem. We now present these theorems to close the section.

### 9.2.3 Stating the theorems

Here we state the structure theorem and the decomposition theorem for skeletal claw-free graphs with $\alpha \geq 4$. First is the refinement of Theorem 8.21 with the new characterizations of strips.

**Theorem 9.4.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- *$G$ admits a 1-join*
- *$G$ is a circular interval graph*
- *$G$ is a composition of linear interval strips, antihat strips, pseudo-line strips, strange strips, and gear strips.*

This theorem is a corollary of Theorem 7.2 from [CS08b], stated in the terms we have defined in this chapter. Of the fifteen types of strips defined in [CS08b], ten correspond to 1-joins and five correspond to the five types of strips we have just defined.

To attack optimization problems on these graphs we will need a corresponding decomposition theorem:

**Theorem 9.5.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- *$G$ admits a 1-join*
- *$G$ is a circular interval graph or a line graph*
- *$G$ admits a canonical interval 2-join, an antihat 2-join, a pseudo-line 2-join, a strange 2-join, or a gear 2-join.*

## 9.3 Claw-free graphs with $\alpha \leq 3$

We now describe the structure of skeletal claw-free graphs containing no stable set of size four. We must consider two separate cases, depending on whether or not $\chi(\overline{G}) \leq 3$.

### 9.3.1 Graphs not covered by three cliques

We already have all the tools we need to describe claw-free graphs with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$.

**Theorem 9.6.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$. Then one of the following applies:*

- *$G$ admits a 1-join*
- *$G$ is a circular interval graph*
- *$G$ is a composition of linear interval strips, antihat strips, pseudo-line strips, strange strips, and gear strips*
- *$G$ is an antiprismatic thickening*

- *$G$ is an icosahedral thickening.*

From this structure theorem we get a corresponding decomposition theorem.

**Theorem 9.7.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$. Then one of the following applies:*

- *$G$ admits a 1-join*
- *$G$ is a circular interval graph or a line graph*
- *$G$ admits a canonical interval 2-join, an antihat 2-join, a pseudo-line 2-join, a strange 2-join, or a gear 2-join*
- *$G$ is an antiprismatic thickening*
- *$G$ is an icosahedral thickening.*

### 9.3.2 Three-cliqued graphs

We now turn our attention to claw-free graphs with $\chi(\overline{G}) = 3$. Given cliques $A$, $B$, and $C$ that partition the vertices of $G$, we say that $(G, A, B, C)$ is a *three-cliqued claw-free graph*. We also sometimes just call $G$ a three-cliqued claw-free graph without specifying a 3-colouring of $\overline{G}$.

A skeletal three-cliqued claw-free graph either admits a *worn hex-join* or belongs to one of several base classes. We first define these base classes, then we explain how to compose them. For a worn hex-join to result in a skeletal graph, most base graphs must be *weakly skeletal*:

**Definition 9.8.** *Let $(X, Y)$ be a nonskeletal homogeneous pair of cliques in a three-cliqued graph $(G, A, B, C)$. Then $(X, Y)$ is* weakly skeletal *if at least one of $X$ or $Y$ intersects more than one of $A$, $B$, and $C$. We say that $(G, A, B, C)$ is* weakly skeletal *if every nonskeletal homogeneous pair of cliques is weakly skeletal.*

The following fact justifies our focus on weakly skeletal base graphs; we leave the simple proof to the reader:

**Observation 9.9.** *Let $(X, Y)$ be a non-weakly-skeletal homogeneous pair of cliques in a three-cliqued graph $(G_1, A_1, B_1, C_1)$. If $(G, A, B, C)$ is a worn hex-join of $(G_1, A_1, B_1, C_1)$ and any three-cliqued graph $(G_2, A_2, B_2, C_2)$, then $(X, Y)$ is not skeletal in $(G, A, B, C)$. In particular, $(G, A, B, C)$ is not skeletal.*

#### Base classes of three-cliqued claw-free graphs

The first four classes we define correspond to thickenings of members of the classes $\mathcal{TC}_1, \ldots, \mathcal{TC}_4$ as defined by Chudnovsky and Seymour [CS08b]. Recall once again that a triad is a stable set of size three.

- **A type of line graph**. Let $H$ be a multigraph with pairwise nonadjacent vertices $a, b, c$ such that each of $a, b, c$ has at least three neighbours, and such that every edge of $H$ has an endpoint in $\{a, b, c\}$. We further insist that for distinct $u, v \in \{a, b, c\}$ there is at most one vertex outside $\{a, b, c\}$ that sees $u$ but not $v$. Let $G = L(H)$, and let cliques $A$, $B$, and $C$ in $G$ correspond to the edges incident to $a$, $b$, and $c$ respectively in $H$. Then $(G, A, B, C)$ is a three-cliqued claw-free graph. Let $\mathcal{TTC}_1$ be the set of all such three-cliqued graphs such that every vertex is in a triad, with the added condition of being weakly skeletal.

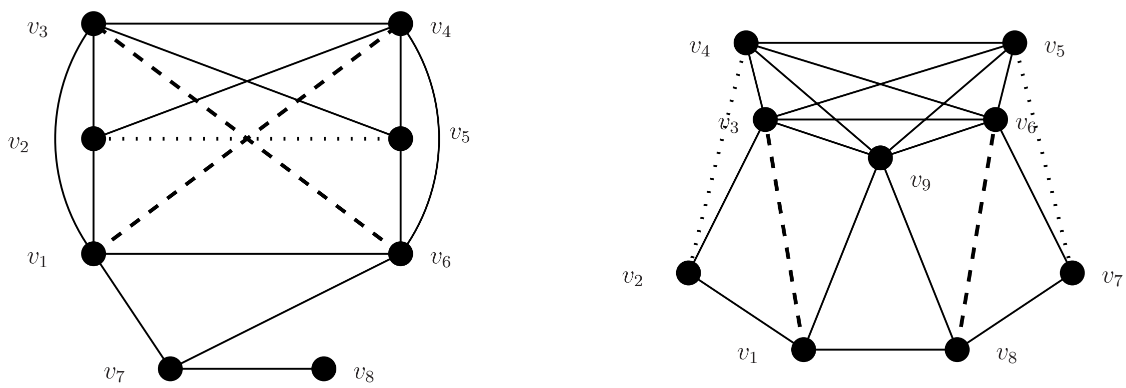


Figure 9.4: The graphs underlying exceptional thickenings in $\mathcal{TC}_5'$ (left) and $\mathcal{TC}_5''$ (right). Solid, dashed, and dotted lines represent adjacent vertices, edges in $M$, and unspecified adjacency respectively. All other pairs are nonadjacent.

- **Long circular interval graphs.** Let $(G, A, B, C)$ be a three-cliqued long circular interval graph with a circular interval representation such that each of $A$, $B$, $C$ is a set of consecutive vertices in circular order. Let $\mathcal{TTC}_2$ be the set of all such graphs such that every vertex is in a triad.

- **Antihat thickenings.** Let $G$ be an antihat thickening, and let $A, B, C$, and $X$ be as they are in the definition of $G$. Let $A' = A \setminus X$ and define $B'$ and $C'$ accordingly. Then $(G - X, I(A'), I(B'), I(C'))$ is a three-cliqued claw-free graph. Let $\mathcal{TTC}_3$ be the class of all such three-cliqued graphs with the added condition of being weakly skeletal.

- **Antiprismatic thickenings.** Let $(G, A, B, C)$ be a three-cliqued antiprismatic graph, and let $(G', I(A), I(B), I(C))$ be a proper thickening of $G$. Let $\mathcal{TTC}_4$ be the class of all such graphs $(G', I(A), I(B), I(C))$.

The final two exceptional cases correspond to thickenings of graphs in Chudnovsky and Seymour's class $\mathcal{TC}_5$ [CS08b].

- **Exception I.** Let $G$ be a graph on vertices $v_1, \ldots, v_8$ with adjacency as follows: $v_1$ is adjacent to $v_2, v_3, v_6, v_7$; $v_2$ is adjacent to $v_3, v_4$; $v_3$ is adjacent to $v_4, v_5$; $v_4$ is adjacent to $v_5, v_6$; $v_5$ is adjacent to $v_6$; $v_6$ and $v_8$ are adjacent to $v_7$; $v_2$ may or may not be adjaecnt to $v_5$. There are no other edges. Now let $M$ be a matching containing $v_1v_4$, $v_3v_6$, and possibly $v_2v_5$. Let $X \subseteq \{v_3, v_4\}$. Let $G'$ be a thickening of $(G \cup M) - X$ under $M$ (see Figure 9.4). Then $(G', I(\{v_1, v_2, v_3\}), I(\{v_4, v_5, v_6\}), I(\{v_7, v_8\}))$ is a three-cliqued claw-free graph. Let $\mathcal{TTC}_5$ be the set of all such graphs with the added condition of being weakly skeletal.

- **Exception II.** Let $G$ be a graph on vertices $v_1, \ldots, v_9$ with the following structure. Let $A = \{v_1, v_2\}$, $B = \{v_7, v_8\}$, and $C = \{v_3, v_4, v_5, v_6, v_9\}$ be cliques. Let $v_1$ be adjacent to $v_3$, $v_8$, and $v_9$. Let $v_8$ be adjacent to $v_6$ and $v_9$. Let $v_2$ be adjacent to $v_3$ and possibly $v_4$. Let $v_7$ be adjacent to $v_6$ and possibly $v_5$. Now let $M$ be a matching in $G$ containing $v_1v_3$ and $v_6v_8$, as well as possibly $v_2v_4$ and $v_5v_7$ (see Figure 9.4). Let $X$ be a subset of $\{v_3, v_4, v_5, v_6\}$ such that:

- $v_2$ and $v_7$ each have a neighbour in $C \setminus X$.
- If $X$ contains neither $v_4$ nor $v_5$ then $v_2$ is adjacent to $v_4$ and $v_7$ is adjacent to $v_5$.

Again we insist that every vertex of $(G - M) - X$ is in a triad. Let $G'$ be a thickening of $G - X$ under $M$. Then $(G', I(A), I(B), I(C \setminus X))$ is a three-cliqued claw-free graph. Let $\mathcal{TTC}_6$ be the set of all such graphs with the added condition of being weakly skeletal.

We allow permutations of the sets $A, B, C$ for any of these classes, so if $(G, A, B, C)$ is in $\mathcal{TTC}_i$ for some $1 \leq i \leq 5$ and $\{A', B', C'\} = \{A, B, C\}$, then $(G, A', B', C')$ is also in $\mathcal{TTC}_i$.

Having described the building blocks for three-cliqued claw-free graphs, we now move on to how they can be combined.

### Worn hex-chains and worn hex-joins

We build three-cliqued claw-free graphs from the base classes we just defined, using a single composition operation: worn hex-chains. First we define a simpler version, *hex-chains*, that we prefer to use whenever possible. These composition operations are closely related to circular interval graphs. For $k \geq 1$ and $1 \leq i \leq k$ let $(G_i, A_i, B_i, C_i)$ be a three-cliqued graph. Suppose we construct $G$ from the disjoint union of $G_i$, $1 \leq i \leq k$ by adding edges as follows:

- Let $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$, $C = \cup_{i=1}^k C_i$ be cliques.
- For $1 \leq i, j \leq k$, make $A_i \cup B_j$, $B_i \cup C_j$, and $C_i \cup A_j$ cliques precisely if $j < i$.

Then we call the sequence $(G_i, A_i, B_i, C_i)$ a *hex-chain* for $(G, A, B, C)$. If $k = 2$, we say that $(G, A, B, C)$ is a *hex-join* of $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$.

Suppose $(G, A, B, C)$ admits a hex-chain into $(G_i, A_i, B_i, C_i)$ for $1 \leq i \leq k$, and let $J \subseteq \{1, \ldots, k\}$ be the set of terms in $\mathcal{TTC}_4$. Further suppose that $(G', A, B, C)$ is constructed from $(G, A, B, C)$ by adding edges such that (i) every added edge is between a vertex in $G_i$ and a vertex in $G_j$ for $i \neq j$ both in $J$, and (ii) no endpoint of an added edge is in a triad in $G$. Then we say that $(G', A, B, C)$ is a *worn hex-chain* of $(G_i, A_i, B_i, C_i)$ for $1 \leq i \leq k$. If $k = 2$ we call it a *worn hex-join*.

To see why hex-chains generalize circular interval graphs, consider the three-cliqued circular interval graph $(G_c, \{a_1, \ldots, a_k\}, \{b_1, \ldots, b_k\}, \{c_1, \ldots, c_k\})$ such that $a_i$ and $b_j$ (resp. $b_i$ and $c_j$; $c_i$ and $a_j$) are adjacent precisely if $j < i$. Then the hex-chain $(G, A, B, C)$ of $(G_i, A_i, B_i, C_i)$ is the result of taking the disjoint union of these graphs and joining those cliques whose corresponding vertices in $G_c$ are adjacent. Thus $(G, A, B, C)$ is also a generalization of a fuzzy circular interval graph, where instead of allowing homogeneous pairs of cliques, we allow certain homogeneous triples of cliques that induce claw-free graphs.

Our structure theorem for skeletal three-cliqued claw-free graphs tells us that aside from the complication of worn hex-chains, the minimal homogeneous triples of cliques are in the base classes we just defined:

**Theorem 9.10.** *Let $(G, A, B, C)$ be a skeletal three-cliqued claw-free graph. Then for some $k \geq 1$, $(G, A, B, C)$ admits a worn hex-chain into terms $(G_i, A_i, B_i, C_i)$ for $1 \leq i \leq k$, each of which is in one of $\mathcal{TTC}_1, \ldots, \mathcal{TTC}_6$.*

This theorem is a straightforward weakening of Chudnovsky and Seymour's more precise structure theorem for three-cliqued claw-free trigraphs (4.1 in [CS08b]). Notice that if $(G, A, B, C)$ is a worn hex-join of $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$ and $T$ is a triad in $G_1$, then every vertex in $G_2$ has exactly two neighbours in $T$. This leads to a useful observation about graphs in $\mathcal{TTC}_4$.

**Lemma 9.11.** *If $(G, A, B, C)$ is a worn hex-join of $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$, both of which are in $\mathcal{TTC}_4$, then $(G, A, B, C)$ is in $\mathcal{TTC}_4$.*

Observe that for the worn hex-chain in the previous structure theorem, $(G \setminus G_1, A \setminus A_1, B \setminus B_1, C \setminus C_1)$ admits a worn hex-chain into $(G_i, A_i, B_i, C_i)$ for $2 \leq i \leq k$. If every term is in $\mathcal{TTC}_4$ then $(G, A, B, C)$ is in $\mathcal{TTC}_4$, otherwise we can assume that $(G_1, A_1, B_1, C_1)$ is in one of $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5, \mathcal{TTC}_6$. This leads us to the desired decomposition theorem:

**Theorem 9.12.** *Any skeletal three-cliqued claw-free graph $(G, A, B, C)$ not in $\mathcal{TTC}_4$ admits a hex-join into terms $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$, where $(G_1, A_1, B_1, C_1)$ is in one of $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5, \mathcal{T}$*

Now that we know the structure of every skeletal claw-free graph, we look at how we can colour them.

# Chapter 10

# Proving the Main Conjecture for Claw-free Graphs

In this chapter we will prove two results:

**Theorem 10.1.** *For any claw-free graph $G$, $\chi(G) \leq \gamma(G)$.*

**Theorem 10.2.** *For any three-cliqued claw-free graph $G$, $\chi(G) \leq \gamma_l(G)$.*

That is, the Main Conjecture holds for all claw-free graphs, and the Local Strengthening holds for all three-cliqued claw-free graphs. Our proofs lead to polynomial-time algorithms for finding a colouring that satisfies the bound.

The proof of the Main Conjecture for line graphs, which we presented in Chapter 4, illustrates our general approach to these two theorems. We assume that $G$ is a minimum counterexample and use the decomposition theorems from the previous chapter. If possible, we remove a stable set $S$ from $G$ such that $\gamma_l(G-S) < \gamma_l(G)$ or $\gamma(G-S) < \gamma(G)$. This contradicts the minimality of $G$. If this is not possible, we use the structure of the graph to prove that $\chi(G) \leq \gamma_l(G)$ or $\chi(G) \leq \gamma(G)$. Actually the situation is sometimes slightly more complicated; when dealing with strips we will ensure that a carefully chosen invariant related to $\gamma_l(G)$ and $\gamma(G)$ behaves appropriately.

## 10.1 The importance of being skeletal

In Chapter 6 we explained that skeletal claw-free graphs are easier to describe and colour than general claw-free graphs. In the previous chapter we gave a simplified version of Chudnovsky and Seymour's structure theorems by describing only skeletal claw-free graphs. The ease with which we can describe skeletal claw-free graphs certainly helps us with the colouring, but skeletal graphs are also simple to colour for a very specific reason which we now explain.

Our approach to colouring often involves removing a stable set $S$ from a supposedly minimum counterexample $G$ and looking at what happens to a set $C$ of vertices. We generally need to confirm that when we remove $S$ from $G$, $\max_{v\in C}(d(v) + \omega(v))$ drops by two. We can easily insist that $S$ be a maximal stable set, so $d(v) + \omega(v)$ drops by at least one for every vertex in $C - S$. Thus we only need to worry about vertices in $C$ maximizing $d(v) + \omega(v)$. In particular, if there is a vertex $v$ in $C$ whose closed neighbourhood properly contains the closed neighbourhood of $v'$, we can safely disregard $v'$ in our analysis. In this case we say that $v$ *trumps* $v'$.

Now consider the vertices in a homogeneous pair of cliques $(A, B)$. If $(A, B)$ is skeletal, then all edges between $A$ and $B$ are contained in a clique $\Omega(A, B)$. We can make several simple observations:

1. Any vertex in $A \setminus \Omega(A, B)$ is trumped by any vertex in $A \cap \Omega(A, B)$.
2. Removing a vertex from $A \cap \Omega(A, B)$ lowers $d(v)$ for any $v \in A \cup \Omega(A, B)$.
3. Removing a vertex from $A \cap \Omega(A, B)$ lowers $\omega(v)$ for any $v \in A$.
4. Removing a vertex from $A \cap \Omega(A, B)$ and a vertex from $B \setminus \Omega(A, B)$ lowers $d(v)$ by two for any $v \in B \cap \Omega(A, B)$, and lowers $\omega(v)$ for any $v \in B \setminus \Omega(A, B)$.

These facts will be of great use to us throughout this chapter.

Before moving on we present some terminology and notation that will aid us. For a set of vertices $S$ we define $\Delta(S)$ as $\max_{v \in S} d(v)$. Likewise we define $\omega(S)$ as $\max_{v \in S} \omega(v)$ and $\gamma_l(S)$ as $\max_{v \in S} \gamma_l(v)$. For convenience, when talking about a thickening we often use $\Omega(v_i v_j)$ to denote $\Omega(I(v_i), I(v_j))$.

We begin by proving the Local Strengthening for antiprismatic thickenings. This serves as an illustrative warm-up.

## 10.2 A warm-up: Antiprismatic thickenings

In Section 2.6 we explained that antiprismatic thickenings motivated the Local Strengthening of the Main Conjecture. The easiest way to prove the Main Conjecture for antiprismatic thickenings is by proving the Local Strengthening. Recall Lemma 9.2, which states that skeletal antiprismatic thickenings are precisely the proper thickenings of antiprismatic graphs. Also note that antiprismatic thickenings are a hereditary class of graphs.

**Theorem 10.3.** *Let $G$ be an antiprismatic thickening. Then $\chi(G) \leq \gamma_l(G)$.*

*Proof of Theorem 10.3.* Let $G$ be a minimum counterexample to the theorem. By Corollary 9.3, we can assume that $G$ is a proper thickening of an antihat graph $H$. By Theorem 2.15 we know that $\alpha(G) = 3$. Let $T$ be any triad in $G$; we claim that $\gamma_l(G - T) < \gamma_l(G)$, contradicting the minimality of $G$.

Suppose $v \notin T$ has only one neighbour in $T$. Lemma 9.2 tells us that $G$ is a proper thickening of some antiprismatic graph $H$. By the definition of antiprismatic graphs, $v$ has some twin $u \in T$. Thus $T$ intersects every maximal clique containing $v$, so if $d(v)$ does not drop by two when $T$ is removed, $\omega(v)$ drops by one and $d(v)$ drops by one. Therefore $\gamma_l(G - T) < \gamma_l(G)$, contradicting the minimality of $G$. The theorem follows. $\square$

## 10.3 Three-cliqued claw-free graphs

To bound the chromatic number of three-cliqued claw-free graphs, we generalize our approach of removing triads, which we just used to deal with antiprismatic thickenings. Three-cliqued claw-free graphs have stability number three unless they are antiprismatic. We will proceed by removing a triad with nice properties whenever possible.

**Definition 10.4.** *Let $T$ be a triad in a graph $G$. If every vertex $v$ in $G - T$ has two neighbours in $T$ or a twin in $T$ or is trumped by a vertex in $T$, then we say that $T$ is a* good triad.

Good triads make very good candidates for a colour class when we are trying to $\gamma_l$-colour a graph:

**Lemma 10.5.** *Let $T$ be a good triad in a graph $G$. Then $\gamma_l(G-T) < \gamma_l(G)$.*

**Corollary 10.6.** *No minimum counterexample to Theorem 10.2 contains a good triad.*

Furthermore, good triads behave nicely with respect to worn hex-joins:

**Observation 10.7.** *Suppose that a three-cliqued claw-free graph $(G, A, B, C)$ admits a worn hex-join into $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$. If $T$ is a good triad in $G_1$, then it is also a good triad in $G$.*

Let $G$ be a minimum counterexample to Theorem 10.2. Then $G$ is skeletal and is not an antiprismatic thickening. So Theorem 9.12 implies that $G$ admits a worn hex-join into $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$, where $(G_1, A_1, B_1, C_1)$ is in $\mathcal{TTC}_1$, $\mathcal{TTC}_2$, $\mathcal{TTC}_3$, $\mathcal{TTC}_5$, or $\mathcal{TTC}_6$. We deal with these three possibilities individually.

First note that $G_2$ may be empty, but this does not affect our approach. Also note that since $G$ is skeletal, $(G_1, A_1, B_1, C_1)$ is weakly skeletal.

### 10.3.1 Five classes to consider

We now prove a set of lemmas that together imply Theorem 10.2, dealing with the easier cases first.

#### Long circular interval graphs ($\mathcal{TTC}_2$)

**Lemma 10.8.** *Any three-cliqued graph $(G_1, A_1, B_1, C_1)$ in $\mathcal{TTC}_2$ contains a good triad.*

*Proof.* Suppose to the contrary that $(G_1, A_1, B_1, C_1)$ is in $\mathcal{TTC}_2$, and call the vertices of $G$ $\{a_1, \ldots, a_i, b_1, \ldots, b_j, c_i, \ldots$ in circular order.

We can find a triad $T$ containing $a_1$ by adding $b_p$ for the minimum $p$ such that $b_p$ does not see $a_1$, then adding $c_q$ for the minimum $q$ such that $b_p$ does not see $c_q$. The triad $T$ exists since $a_1$ is in a triad and it follows from the structure of circular interval graphs that $a_1$ and $b_p$ are in a triad together. If some vertex in $(A \setminus \{a_1\}) \cup \{b_x \mid x < p\}$ does not see both $a_1$ and $b_p$, then we are in a degenerate case where $G_1$ is a linear interval graph, and the vertex in question is a twin of $a_1$ or $b_p$, or it is trumped by $a_1$ or $b_p$. The same applies to every vertex in $\{b_x \mid x > p\} \cup \{c_y \mid y < q\}$: each vertex has two neighbours in $T$ or a twin in $T$ or is trumped by a vertex in $T$. Similarly, if some vertex $v$ in $\{c_l \mid l > q\}$ has only one neighbour in $T$ then it has no neighbours in $A$, hence it is trumped by or is a twin of $c_q$. Thus $T$ is a good triad. $\square$

#### Antihat thickenings ($\mathcal{TTC}_3$)

**Lemma 10.9.** *Any three-cliqued graph $(G_1, A_1, B_1, C_1)$ in $\mathcal{TTC}_3$ contains a good triad.*

*Proof.* Let $T$ be a triad consisting of a vertex $a$ of $I(a_0)$ and vertices $b$ in $I(B \setminus \{b_0\})$ and $c$ in $I(C)$ respectively, following the definition of an antihat thickening. Furthermore if $b$ and $c$ are in $I(b_i)$ and $I(c_i)$ respectively, we insist that $T$ intersects $\Omega(b_i c_i)$. We insist that if $I(a_0) \cap \Omega(a_0 b_0)$ is nonempty, then $T$ intersects it. It is easy to confirm from the structure of an antihat thickening that $T$ exists and is a good triad. $\square$

**Exception I ($\mathcal{TTC}_5$)**

**Lemma 10.10.** *Any three-cliqued graph $(G_1, A_1, B_1, C_1)$ in $\mathcal{TTC}_5$ contains a good triad.*

*Proof.* Let $T$ be a triad including one vertex in each of $I(v_7)$, $I(v_2)$, and $I(v_5)$, such that $T$ intersects $\Omega(v_2v_5)$ if it is not empty. It is easy to confirm that $T$ is a good triad, since any vertex in $I(v_2) \setminus \Omega(v_2v_5)$ is trumped by a vertex in $I(v_3)$ if $v_3 \notin X$, and is trumped by a vertex in $I(v_6)$ if $v_3 \notin X$. Similarly any vertex in $I(v_2) \setminus \Omega(v_2v_5)$ is trumped (see Figure 9.4). □

**Exception II ($\mathcal{TTC}_6$)**

**Lemma 10.11.** *Any three-cliqued graph $(G_1, A_1, B_1, C_1)$ in $\mathcal{TTC}_6$ contains a good triad.*

*Proof.* Let $T$ be a triad including one vertex in each of $I(v_2)$, $I(v_7)$, and $I(v_9)$, such that $T$ intersects $\Omega(v_2v_4)$ if it is not empty, and intersects $\Omega(v_5v_7)$ if it is not empty. It is easy to confirm that $T$ is a good triad (see Figure 9.4). □

**A type of line graph ($\mathcal{TTC}_1$)**

We now prove the necessary lemma for $G_1$ in $\mathcal{TTC}_1$. This is by far the most difficult case. We make extensive use of the fact that line graphs of bipartite multigraphs are perfect.

**Lemma 10.12.** *Let $(G, A, B, C)$ be a minimum counterexample to Theorem 10.2 and suppose it admits a worn hex-join into $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$. Then $(G_1, A_1, B_1, C_1)$ is not in $\mathcal{TTC}_1$.*

*Proof.* Suppose $(G_1, A_1, B_1, C_1)$ is in $\mathcal{TTC}_1$. Then $G_1$ is the line graph of some bipartite multigraph $H$ which has a stable set $\{a, b, c\}$ corresponding to $A_1$, $B_1$, and $C_1$. Assume without loss of generality that $|C_1| \leq |B_1| \leq |A_1|$. We call the other vertices of $H$ *centres*. Depending on the structure of $G_1$ we will take one of two actions:

1. Remove a triad from $G_1$, lowering $\gamma_l(G)$.
2. Remove edges from $G_1$ without changing $\chi(G)$.

Every vertex of $G_1$ is in a triad. If there are only three centres then removing any triad $T$ will lower $\gamma_l(G)$ since every vertex in $G_1 - T$ will have two neighbours or a twin in $T$ – this can be confirmed easily since the graph underlying $H$ will be a subgraph of $K_{3,3}$. So there are at least four centres. Call the four centres of highest degree $w$, $x$, $y$, and $z$ such that $d(w) \geq d(x) \geq d(y) \geq d(z)$.

For any centre $s$, denote by $A_s$ the clique corresponding to the edges of $H$ between $a$ and $s$. Define $B_s$ and $C_s$ accordingly. Denote $A_s \cup B_s \cup C_s$ by $X_s$. We now consider, for some vertex $v \in A_s$, what cliques of size $\omega(v)$ can contain $v$. By the structure of a hex-join[1], observe that such a clique must be one of:

- A clique in $G_1$ intersecting all of $A_1, B_1, C_1$. Specifically, $A_s \cup B_s \cup C_s = X_s$.
- A clique in $A_1 \cup B_1 \cup A_2$ containing all of $A_2$. Specifically, $A_2 \cup A_s \cup B_s$.
- A clique in $A_1 \cup C_1 \cup C_2$ containing all of $C_2$. Specifically, $C_2 \cup A_s \cup C_s$.

[1]As we noted at the end of the previous chapter, the worn hex-join is a hex-join because every vertex of $G_1$ is in a triad.

- A clique in $A_1 \cup A_2 \cup C_2$ containing all of $A_1$. Such a clique has size at least $|A_1| + \max\{|A_2|, |C_2|\}$.

Note also that the closed neighbourhood of $v$ is $A_1 \cup X_s \cup A_2 \cup C_2$. We can make similar observations about the cliques of size $\omega(v)$ when $v$ is in $A_1 \setminus A_s$ or $B_1$ or $C_1$. These observations tell us when removing a triad $T$ lowers $\omega(v)$ and therefore $\gamma_l(v)$.

Note that at most two centres have degree $\geq d(a)$, since there are at least four centres and the sum of their degrees is $d(a) + d(b) + d(c)$. Suppose there are at most three centres with degree $\geq d(c)$. Then we can hit them all with a triad $T$ by the definition of $\mathcal{TTC}_1$. We will now show that removing $T$ will lower $\gamma_l(v)$ for all $v \in G_1$. This triad $T$ will hit $A_1$, $B_1$, and $C_1$. Any vertex $v$ in $G_1$ without two neighbours or a twin in $T$ will correspond to an edge in $H$ incident to some centre $s$, where $d(s) < d(c)$. By our above observations about cliques of size $\omega(v)$, we can see that since $|X_s| < |C_1| \leq |B_1| \leq |A_1|$, any clique of size $\omega(v)$ containing $v$ must contain one of $C_1$, $B_1$, or $A_1$. Therefore such a clique intersects $T$, so removing $T$ lowers $\omega(v)$ and also $\gamma_l(v)$. This contradicts the minimality of $G$, so we can assume that there are at least four centres of degree $\geq |C|$, i.e. $d(z) \geq d(c)$. We now consider several cases.

**Case 1:** $d(w) \geq d(a)$ and $c$ sees $w$.

Since $d(w) \geq d(a)$ it follows that $d(x) + d(y) + d(z) \leq |B_1| + |C_1|$, and so $d(x) + d(y) \leq |B_1|$ and $|C_1| \leq d(z) \leq \frac{1}{3}(|B_1| + |C_1|)$. Therefore $2|C_1| \leq 2d(z) \leq d(x) + d(y) \leq |B_1|$. Take a triad $T$ that hits $X_w$, $X_x$, and $X_y$, and consider a vertex $v$ for which $\omega(v)$ does not drop when $T$ is removed. Clearly $v$ is not in $X_w \cup X_x \cup X_y$, so it is in $X_s$ for some centre $s$ with $d(s) \leq d(z) \leq \frac{1}{2}|B_1|$. Since $|X_s| < |B_1|$ and $\omega(v)$ does not drop, $v$ must be in $C_s$. Take some $u \in C_w$. We will show that $d(u) + \omega(u) > d(v) + \omega(v)$, which implies that $\gamma_l(G - T) < \gamma_l(T)$.

Clearly $u$ has at least $|A_1| - |C_1|$ neighbours in $G_1 - C_1$. But $v$ has at most $\frac{1}{2}|B_1| - 1$ neighbours in $G_1 - C_1$. Therefore $d(u) > d(v) + \frac{1}{2}|A_1| - |C_1|$. Recall the structure of maximal cliques containing $u$ and $v$. If $\omega(v) > \omega(u)$ then either $|C_s| + |A_s| > \max\{|C_w| + |A_w|, |C_1|\}$ or $|C_s| + |B_s| > \max\{|C_w| + |B_w|, |C_1|\}$. But in this case $\omega(v) \leq \omega(u) + d(s) - |C_1| \leq \omega(u) + \frac{1}{2}|A_1| - |C_1|$. It follows that $d(v) + \omega(v) < d(u) + \omega(u)$, completing the case.

**Case 2:** $d(w) \geq d(a)$ and $c$ does not see $w$.

Make the subgraph $G'$ of $G$ by removing all edges between $C_1$ and $G_1 \setminus C_1$ – observe that $G'$ is claw-free and three-cliqued. Further observe that because $H$ has at least four centres, if $G' = G$ then $(A_1, B_1)$ is a nonskeletal homogeneous pair of cliques in $G$, a contradiction. Thus $G'$ is a proper subgraph of $G$. We claim that $\chi(G') = \chi(G)$, contradicting the minimality of $G$. Denote by $G_1'$ the subgraph of $G'$ induced on the vertices of $G_1$.

Take a $\chi(G')$-colouring $\mathcal{C}'$ of $G'$. We will rearrange the colour classes of $\mathcal{C}'$ on $G_1'$ to reach a proper colouring of $G_1$. Denote by $t$ the number of triad colour classes in $\mathcal{C}'$ restricted to $G_1'$. Denote by $d_{AB}$, $d_{AC}$, and $d_{BC}$ the number of diads (i.e. colour classes of size two) in $\mathcal{C}'$ restricted to $G_1'$ intersecting $A_1$ and $B_1$, $A_1$ and $C_1$, and $B_1$ and $C_1$ respectively. It suffices to show that we can pack the appropriate disjoint stable sets into $G_1$. That is, we want to find $t$ triads in $G_1$, $d_{AB}$ diads intersecting $A_1$ and $B_1$, $d_{AC}$ diads intersecting $A_1$ and $C_1$, and $d_{BC}$ diads intersecting $B_1$ and $C_1$, such that all of these stable sets are disjoint.

We begin with $|A_1| + |B_1| - d(w)$ diads intersecting $A_1$ and $B_1$. Since $G[A_1 \cup B_1]$ is cobipartite, these diads hit every vertex of $(A_1 \cup B_1) \setminus X_w$. Observe that $|A_1| + |B_1| - d(w) \geq t + d_{AB}$. So we want to extend some of the diads to triads. We can actually extend $|C_1|$ of them. To see this, note that there are at least three triads of degree $\geq |C_1|$ other than $w$, so every vertex in $C_1$ has at least $C_1$ non-neighbours in $(A_1 \cup B_1) \setminus X_w$. So we have $|A_1| + |B_1| - d(w) - |C_1|$ disjoint diads intersecting $A$ and $B$ and a further $|C_1|$ disjoint triads.

Thus it is clear that we can find the desired disjoint stable sets, beginning with the diads intersecting $A$ and $B$. When picking our $d_{AC} + d_{BC}$ remaining diads we take a vertex in $X_w$ not intersecting an $AB$ diad whenever possible. Once we have found the necessary diads, we have enough $AB$ diads remaining so that we can extend them to triads. These stable sets give us a $\chi(G')$-colouring of $G$, contradicting the minimality of $G$.

**Case 3:** $d(w) < d(a)$.

As in the previous case, we remove edges from $G_1$ without introducing a claw or changing the chromatic number of $G$. There is at most one clique $X$ in $G[B_1 \cup C_1]$ of size greater than $|B_1|$. If $X$ exists, construct $G'$ from $G$ by removing all edges from $G_1$ except those within $A_1$, $B_1$, $C_1$, and $X$. If such an $X$ does not exist, set $X$ as $B_1$ and construct $G'$ from $G$ by removing all edges from $G_1$ except those within $A_1$, $B_1$, and $C_1$. It is easy to confirm that $G'$ is claw-free and a proper subgraph of $G$. We will show that $\chi(G') = \chi(G)$, contradicting the minimality of $G$.

We claim that there is an $\omega(G_1)$-colouring of $G_1$ using $|B_1| + |C_1| - |X|$ triads. To see this, we remove $|A_1| - |X|$ vertices from $A_1$ one at a time, always taking one from the largest clique $X_s$ that still has a vertex in $A_1$. If after removing $k$ vertices we have disjoint $X_s$ and $X_{s'}$ of size $|A_1| - k$, then we have $|A_1| + |B_1| + |C_1| \geq k + 2(|A_1| - k) + 2|C_1|$, contradicting the fact that there are at least four centres of degree $\geq |C_1|$ in $H$ and $|B_1| \geq |A_1| - k$. Thus we can see that we reach a perfect graph on $|X| + |B_1| + |C_1|$ vertices with clique number $|X|$. In an $|X|$-colouring of this graph every colour class intersects both $A_1$ and $X$, thus the colouring uses exactly $|B_1| + |C_1| - |X|$ triads. The other colour classes are diads intersecting $X$ and $A_1$. Thus as in the previous case, we can rearrange the colour classes of a $\chi(G')$-colouring $\mathcal{C}'$ of $G'$ to construct a $\chi(G')$-colouring of $G$. □

### 10.3.2 Completing the proof

We now combine our lemmas to prove Theorem 10.2.

*Proof of Theorem 10.2.* Let $(G, A, B, C)$ be a minimum counterexample to the theorem. Then $G$ is skeletal and is not an antiprismatic thickening. Combining Theorem 9.12 with Lemma 9.11 tells us that $(G, A, B, C)$ admits a worn hex-join into $(G_1, A_1, B_1, C_1)$ and $(G_2, A_2, B_2, C_2)$ such that $(G_1, A_1, B_1, C_1)$ is in one of $\mathcal{TTC}_1$, $\mathcal{TTC}_2$, or $\mathcal{TTC}_4$. Lemmas 10.12, 10.8, 10.9, 10.10, and 10.11 tell us that $(G_1, A_1, B_1, C_1)$ cannot be in $\mathcal{TTC}_1$, $\mathcal{TTC}_2$, $\mathcal{TTC}_3$, $\mathcal{TTC}_5$ or $\mathcal{TTC}_6$ respectively. Thus $G$ cannot exist, proving the theorem. □

## 10.4 Icosahedral thickenings

Now that we have proved the Local Strengthening for three-cliqued claw-free graphs, we can extend the result to icosahedral thickenings. We do this by removing triads once again, so first we need to consider induced subgraphs of icosahedral thickenings.

**Lemma 10.13.** *Let $G$ be an icosahedral thickening. Then any induced subgraph $G'$ of $G$ is an icosahedral thickening or contains a clique cutset or admits a canonical linear interval 2-join.*

The proof of this lemma is straightforward. We leave it to the end of this section. This lemma allows us to prove the desired result:

**Theorem 10.14.** *Suppose $G$ is an induced subgraph of an icosahedral thickening. Then $\chi(G) \leq \gamma_l(G)$.*

*Proof.* Let $G$ be a minimum counterexample to the theorem. By Lemma 10.13 we know $G$ is an icosahedral thickening or contains a clique cutset or is three-cliqued or admits a canonical interval 2-join. But $G$ is vertex-critical so it cannot contain a clique cutset. Lemma 7.1 and Theorem 10.2 tell us that $G$ is in fact an icosahedral thickening.

First suppose that $G$ is a proper thickening of $G_0$, the icosahedron. We remark that the icosahedron is 4-colourable, so we remove four stable sets with union $X$ containing exactly one vertex in $I(v_i)$ for every vertex $v_i$ of $G_0$. When $X$ is removed, every remaining vertex $v$ in $G$ loses six neighbours (one of which is a twin), and since every maximal clique in $G$ corresponds to a triangle in $G_0$, $\omega(v)$ drops by three. Thus $d(v)+\omega(v)$ drops by nine and it follows that $\gamma_l(G)$ drops by at least four, contradicting the minimality of $G$.

Now suppose that $G$ is a proper thickening of $G_1$ (see Figure 9.1). Again we remove one vertex from each $I(v_i)$, this time for $0 \leq i \leq 10$, again using four stable sets. When we remove the vertices, every remaining vertex loses five neighbours, one of which is a twin. And as with $G_0$, every vertex $v$ of $G$ has $\omega(v)$ drop by three. Thus $\gamma_l(G)$ drops by at least four, contradicting the minimality of $G$.

Finally suppose that $G$ is a thickening of $G_2 \cup M$ under a matching $M$; we know that $M \subseteq \{v_1v_4, v_6v_9\}$. By minimality of $G$, $(I(v_1), I(v_4))$ and $(I(v_6), I(v_9))$ are skeletal homogeneous pairs of cliques. We remove two stable sets with union $X$. One intersects $I(v_1)$, $I(v_4)$, and $I(v_7)$ and intersects $\Omega(v_1v_4)$ if it is not empty. The other intersects $I(v_3)$, $I(v_6)$, and $I(v_9)$ and intersects $\Omega(v_6v_9)$ if it is not empty. These stable sets must exist because neither $I(v_1)\cup I(v_4)$ nor $I(v_6)\cup I(v_9)$ is a clique.

It is straightforward to confirm that $X$ intersects every maximal clique in $G$, so $\omega(v)$ drops by at least one for every $v \in G - X$, thus $\gamma_l(v)$ drops by at least two for any vertex with three neighbours in $X$. Observe that any vertex in $G - X$ with only two neighbours in $X$ must be in $(I(v_1) \cup I(v_4)) \setminus \Omega(v_1v_4)$ or $(I(v_6) \cup I(v_9)) \setminus \Omega(v_1v_4)$. Furthermore, every such vertex has a twin in $X$. Thus we can easily confirm that $\omega(v)$ drops by two for every such vertex. So for any $v$ with only two neighbours in $X$, $\omega(v)$ drops by two. Therefore $\gamma_l(G - X) \leq \gamma_l(G) - 2$, contradicting the minimality of $G$. This completes the proof. □

We now prove Lemma 10.13.

*Proof of Lemma 10.13.* Suppose first that $G$ is a thickening of $G_2 \cup M$ (see Figure 9.1). If $G'$ has $I(v_i)$ nonempty for all $0 \leq i \leq 9$ then clearly $G_2$ is an icosahedral thickening. If $I(v_i)$ is empty for some $i \in \{0, 2, 5, 8\}$ then it is not hard to check that $G'$ is three-cliqued. If $I(v_i)$ is empty for some $i \in \{1, 4, 6, 9\}$ then $G'$ contains a clique cutset. If none of these aforementioned sets $I(v_i)$ is empty but one of $I(v_3)$ and $I(v_7)$ is empty, then $G'$ admits a canonical interval 2-join. For example, if $G'$ is reached from $G$ by deleting $I(v_3)$, then $((I(v_0) \cup I(v_9), I(v_5 \cup I(v_6)), (I(v_1), I(v_4)))$ is a canonical interval 2-join.

Now suppose that $G$ is a thickening of $G_1$. Obviously $G'$ is an icosahedral thickening if $I(v_i)$ is nonempty for all $0 \leq i \leq 10$. If $I(v_i)$ is empty for any $i \in \{2, 4, 6, 8, 10\}$ then the desired result follows from the previous paragraph. If $I(v_0)$ is empty then $G'$ is a circular interval graph. If $I(v_i)$ is empty for some $i \in \{1, 3, 5, 7, 9\}$ then it is easy to see from Figure 9.1 that $G'$ admits a canonical interval 2-join or a clique cutset.

Finally, suppose that $G$ is a thickening of $G_0$. If $G'$ has any $I(v_i)$ empty for $0 \leq i \leq 11$ then the desired result follows from the previous two paragraphs. Otherwise $G'$ is clearly a thickening of $G_0$. This completes the proof. $\square$

## 10.5 Compositions of strips

In Section 7.1.1 we proved Lemma 7.1, which implies that no minimum counterexample to the Local Strengthening or the Main Conjecture admits a canonical interval 2-join. We now extend this approach, proving that no minimum counterexample to the Local Strengthening or the Main Conjecture admits an antihat, strange, or gear 2-join. Like line graphs, pseudo-line 2-joins present some difficulty with the Local Strengthening. However, later in this section we will prove that a minimum counterexample to the Main Conjecture cannot admit a pseudo-line 2-join. This allows us to prove Theorem 10.1 using our decomposition result, Theorem 9.5.

As with the proof for canonical interval 2-joins, we consider a 2-join $((X_1, Y_1), (X_2, Y_2))$ joining $G_1$ and $G_2$, and we let $H_2$ denote $G[V_2 \cup X_1 \cup Y_1]$. For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in $H_2$ containing $v$ and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^j(H_2)$ as $\max_{v \in H_2} \lceil d_G(v) + 1 + \omega'(v) \rceil$ (here the superscript $j$ denotes *2-join*). Observe that $\gamma_l^j(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$.

In light of what we have already done, the following lemma, which is a generalization of Lemma 7.1, deals with antihat 2-joins, strange 2-joins, and gear 2-joins.

**Lemma 10.15.** *Suppose a skeletal claw-free graph $G$ admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l \geq \gamma_l^j(H_2)$, we can find a proper $l$-colouring of $G$.*

We split the proof up unto three lemmas corresponding to antihat 2-joins, strange 2-joins, and gear 2-joins. Our approach in each case is to set up the colouring of $G_1$ so that we can do one of two things. When possible, we colour $G_2$ directly by constructing an auxiliary graph from $G_2$ and appealing to perfection or Theorem 10.2. If that is not possible then we remove stable sets, reducing $\gamma_l^j(H_2)$ each time, until $G_2$ becomes degenerate and we can appeal to a previous result.

**Lemma 10.16.** *Suppose a skeletal claw-free graph $G$ admits an antihat 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l \geq \gamma_l^j(H_2)$, we can find a proper $l$-colouring of $G$.*

*Proof.* Consider a minimum counterexample for some fixed $l$. As in the proof of Lemma 7.1, we can assume that $l = \gamma_l^j(H_2)$. Furthermore if $G_2$ contains a skeletal homogeneous pair of cliques $(A, B)$ then one of $A$ and $B$ is partially but not completely contained in one of $X_2$ or $Y_2$. We denote $G_2 - X_2 - Y_2$ by $Z_2$.

Let $k$ be the number of colours appearing in both $X_1$ and $Y_1$. We begin by making $k$ minimal, as we did in Case 6 of the proof of Lemma 7.1. This minimality of $k$ ensures a bound on $l$, as long as $k \geq 1$. Let vertices $u \in X_1$ and $v \in Y_1$ have the same colour. Then $d(u) + 1 \geq |X_2| + (l - |Y_1| + k)$, since minimality ensures that $u$ has a neighbour in $G_1$ of every colour except possibly those in $Y_1$ not appearing in $X_1$. Similarly, $d(v) + 1 \geq |Y_2| + (l - |X_1| + k)$. Therefore since $\omega'(u)$ and $\omega'(v)$ are at least $|X_1| + |X_2|$ and $|Y_1| + |Y_2|$ respectively, $l \geq |X_2| + \frac{1}{2}(l + k + |X_1| - |Y_1|)$ and $l \geq |Y_2| + \frac{1}{2}(l + k + |Y_1| - |X_1|)$. Consequently $l \geq |X_2| + |Y_2| + k$ if $k > 0$.

Suppose there is a colour class $S$ in $G_1$ hitting $X_1$ but not $Y_1$. Then add to this colour class a stable set $S'$ of size two intersecting $Y_2$ and $Z_2$. By the structure of antihat thickenings, we can assume that $S'$ intersects $I(b_1)$ and $I(c_1)$ without loss of generality. If $\Omega(b_1c_1)$ is nonempty, we insist that $S'$ intersect it.

Note first that every vertex in $I(b_1)\cup I(c_1)$ is trumped or has a twin in $S'$ or has two neighbours in $S'$. Every vertex in $Z_2\setminus I(c_1)$ has two neighbours in $S'$, as does every vertex in $Y_2\setminus I(b_1)$. Every vertex in $X_2$ has a neighbour in $S$ and a neighbour in $S'$ – this neighbour will be in $Y_2$ for vertices in $I(a_1)$, and in $Z_2$ for all other vertices in $X_2$ (recall from the definition of an antihat 2-join that since $I(b_1)$ and $I(c_1)$ are both nonempty, $I(a_1)$ is complete to $I(b_1)$ and anticomplete to $I(c_1)$). Thus $\gamma_l^j(v)$ drops for any $v\in G_2$. Since $S\cup S'$ intersects both $X_1\cup X_2$ and $Y_1\cup Y_2$, $\gamma_l^j(v)$ drops for any $v\in X_1\cup Y_1$. Therefore we remove $S\cup S'$ and lower $\gamma_l^j(H_2)$.

We repeat this approach until either $Y_2\cup Z_2$ is a clique, or all colours in $X_1$ appear in $Y_1$. Suppose we remove $t_1$ stable sets in this way. We then take colour classes of $G_1$ hitting $Y_1$ but not $X_1$, and remove them along with stable sets of size two in $X_2\cup Z_2$, using the symmetric argument to show that $\gamma_l^j(H_2)$ drops each time. We do this until either all colours appearing in $Y_1$ are in $X_1$, or until $X_2\cup Z_2$ is a clique. Let $t_2$ be the number of stable sets we remove in this way, let $S_1$ be the set of all vertices we have removed from $G$, and let $t=t_1+t_2$. Notice that $\gamma_l^j(H_2-S_1)\le\gamma_l^j(H_2)-t$.

Suppose $X_1\setminus S_1$ is empty. Then we can colour $G_2-S_1$ using $l-t$ colours by Theorem 10.2, since $G_2$ is three-cliqued. Since $Y_1\setminus S_1$ is a clique cutset in $G-S_1$, this immediately gives us an $(l-t)$-colouring of $G-S_1$ and therefore an $l$-colouring of $G$. So we can assume $X_1\setminus S_1$ and symmetrically $Y_1\setminus S_1$ are nonempty.

Now suppose every colour hitting $Y_1\setminus S_1$ also hits $X_1\setminus S_1$. Again we $(l-t)$-colour $G_2-S_1$, noting that at most $|X_2|+|Y_2|-t$ colours appear on $(X_2\cup Y_2)\setminus S_2$ because $|(X_2\cup Y_2)\setminus S_2|=|X_2|+|Y_2|-t$. We ensure that no colour hits both $X_1$ and $X_2$, and that no colour hits both $Y_1$ and $Y_2$. This is possible because $l-t>|(X_1\cup X_2)\setminus S_1|$ and $l-t\ge|X_2|+|Y_2|+k-t$, as we proved above. This gives us a proper $(l-t)$-colouring of $G-S_1$, and therefore an $l$-colouring of $G$.

By symmetry, this covers the case in which every colour hitting $X_1\setminus S_1$ also hits $Y_1\setminus S_1$. Thus there is a colour in $X_1$ but not $Y_1$, and one in $Y_1$ but not $X_1$. So our method stopped because both $(Y_2\cup Z_2)\setminus S_1$ and $(X_2\cup Z_2)\setminus S_1$ are cliques.

In this final case, we $(l-t)$-colour $G_2-S_1$ by applying Lemma 7.1 as follows. Notice that $(X_2\setminus S_1, Y_2\setminus S_1)$ is a homogeneous pair of cliques in $G-S_1$. We reduce it to a skeletal homogeneous pair of cliques without changing the chromatic number using Lemma 6.12; the result is a graph $G'$ in which $((X_1\setminus S_1, Y_1\setminus S_1),(X_2\setminus S_1, Y_2\setminus S_1))$ is a canonical interval 2-join. We can therefore apply Lemma 7.1 to find an $(l-t)$-colouring of $G'$. Again using Lemma 6.12, we can construct an $(l-t)$-colouring of $G-S_1$. This immediately gives us an $l$-colouring of $G$, proving the lemma. □

The next case is strange 2-joins; we use a similar approach.

**Lemma 10.17.** *Suppose a skeletal claw-free graph $G$ admits a strange 2-join $((X_1,Y_1),(X_2,Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l\ge\gamma_l^j(H_2)$, we can find a proper $l$-colouring of $G$.*

*Proof.* As in the proof of the previous lemma, assume $\gamma_l^j(H_2)=l$ and let $k$ denote the number of colours appearing in both $X_1$ and $Y_1$. We begin by modifying the colouring of $G_1$ so that $k$ is minimal, so again we can assume that either $k=0$ or $l\ge|X_2|+|Y_2|+k$. Denote $G_2-X_2-Y_2$ by $Z_2$.

Let $t=\min\{|I(a_1)|,|I(c_1)\cap\Omega(c_1,b_3)|,|Y_2|-k\}$. We remove $t$ colours hitting $Y_1$ but not $X_1$. With each colour class we remove a vertex of $I(a_1)$ and a vertex of $I(c_1)\cap\Omega(c_1,b_3)$. Together these vertices form $t$ stable sets; call their union $S_1$. As in the proof of the previous lemma, we now consider our situation depending on the value of $t$. Note that each time we remove a stable set, every vertex in $G_2$ is either trumped or loses two neighbours or loses a twin. It is therefore easy to see that $\gamma_l^j(H_2 - S_1) \leq \gamma_l^j(H_2) - t$.

Suppose $I(a_1)$ is empty. We apply Lemma 7.1 to $(l-t)$-colour $G - S_1$ as follows. First observe that removing $S_1$ turns $((X_1 \backslash S_1, Y_1 \backslash S_1), (X_2 \backslash S_1, Y_2 \backslash S_1))$ into a fuzzy linear interval 2-join, in which $(Z_2 \setminus S_1, Y_2 \setminus S_1)$ is a homogeneous pair of cliques. We reduce this homogeneous pair to a skeletal homogeneous pair of cliques using Lemma 6.12, at which point $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ becomes a canonical linear interval 2-join in a graph $G'$. We can therefore apply Lemma 7.1 to $G'$, since we already have an $(l-t)$-colouring of $G_1 - S_1$, to find an $(l-t)$-colouring of $G'$. Lemma 6.12 tells us that we can use this colouring to construct an $(l-t)$-colouring of $G - S_1$. Combining this with a $t$-colouring of $G[S_1]$ gives us an $l$-colouring of $G$.

Now suppose $I(c_1) \cap \Omega(c_1, b_3)$ is empty but $I(c_1)$ is not empty. To $(l-t)$-colour $G - S_1$, we first remove the vertices of $I(b_3)$, which have become simplicial. Now observe that $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus (I(b_3) \cup S_1))$ is an antihat 2-join. The remaining sets of $G_2$ are $I(a_1)$, $I(a_2)$, $I(b_1)$, $I(b_2)$, $I(c_1)$, and $I(c_2)$. To see the antihat 2-join, we relabel these sets as in the definition of an antihat thickening. Respectively, these sets become $I(a_1)$, $I(a_2)$, $I(b_1)$, $I(b_3)$, $I(c_3)$, and $I(c_1)$ (see Section 9.1.4). We can therefore apply Lemma 10.16 to find an $(l-t)$-colouring of $G - (S_1 \cup I(b_0))$, then replace and colour the simplicial vertices in $I(b_0)$ to get an $(l-t)$-colouring of $G - S_1$. This gives us an $l$-colouring of $G$, completing the case of strange 2-joins. □

The final and most difficult case is that of gear 2-joins.

**Lemma 10.18.** *Suppose a skeletal claw-free graph $G$ admits a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l \geq \gamma_l^j(H_2)$, we can find a proper $l$-colouring of $G$.*

*Proof.* We proceed by induction on $|G|$, taking as our basis the trivial case in which $\min\{|X_1|, |Y_1|\} = 0$; in this case we have a 1-join and the result follows from Theorem 10.2 since gear strips are three-cliqued. So assume both $X_1$ and $Y_1$ are nonempty. Let $Z_2$ denote $G_2 \setminus (X_2 \cup Y_2)$. Again we can let $G$ be a minimum counterexample and assume that $l = \gamma_l^j(H_2)$.

In this case we make $k$, the overlap between $X_1$ and $Y_1$ in the colouring of $G_1$, maximal.

**Case 1:** $k > 0$.

If $k > 0$, we remove a colour class hitting both $X_1$ and $Y_1$, along with one vertex each of $I(v_9)$ and $I(v_{10})$, if they are both nonempty. In this case every vertex of $G_2$ loses a twin or two neighbours. Since we remove a vertex in both $X_1$ and $Y_1$, it is easy to see that $\gamma_l^j(H_2)$ drops. Since removing vertices from $I(v_9)$ and $I(v_{10})$ will not change the fact that we have a gear 2-join, we can proceed by induction, having reduced both $\gamma_l^j(H_2)$ and $l$.

So assume that $I(v_9) \cup I(v_{10})$ is a clique, i.e. one of $I(v_9)$ and $I(v_{10})$ is empty. We do the same thing, but instead we remove a colour class hitting both $X_1$ and $Y_1$, along with a vertex of $I(v_3)$ and a vertex of $I(v_6)$. Clearly $\gamma_l^j(H_2)$ drops as before and we can proceed by induction, since as long as neither $I(v_3)$ nor $I(v_6)$ becomes empty we will still have a gear 2-join.

Suppose $I(v_6)$ becomes empty, and one of $I(v_9)$ and $I(v_{10})$ is empty. By symmetry we can assume that $I(v_9)$ is empty. We are now left with a fuzzy linear interval 2-join. The vertices, in linear order, are $I(v_1)$, $I(v_2)$, $I(v_7)$, $I(v_3)$, $I(v_{10})$, $I(v_8)$, $I(v_4)$, $I(v_5)$, and the possibly nonlinear homogeneous pairs of cliques are $(I(v_7), I(v_8))$ and $(I(v_3) \cup I(v_{10}), I(v_4) \cup I(v_5))$. The reader can confirm this, along with symmetry between $v_9$ and $v_{10}$, by consulting Figure 9.3.

So, as in the proof of the previous two lemmas, we can find our $l$-colouring of $G$ by reducing on these two homogeneous pairs of cliques and invoking Lemma 7.1.

This completes the proof of the lemma when $k > 0$.

**Case 2:** $k = 0$; $l > |X_1| + |Y_1|$.

In this case we remove a colour class hitting neither $X_1$ nor $Y_1$, along with a stable set of size three in $G_2$. Call their union $S$. If $I(v_{10})$ is nonempty, we remove a vertex of $I(v_{10})$ along with on vertex each of $I(v_1)$ and $I(v_4)$. Every vertex in $G_2$ loses a twin or two neighbours, so it is easy to confirm that $\gamma_l^j(H_2)$ drops. Thus we can proceed by induction, provided that both $I(v_1)$ and $I(v_4)$ are still nonempty.

If $I(v_1)$ and $I(v_4)$ are both empty, then we extend the colouring of $G_1$ to an $l$-colouring of $G_1 \cup I(v_2) \cup I(v_5)$. We then note that $((I(v_2) \cup I(v_{10}) \setminus S, I(v_5) \cup I(v_{10}) \setminus S), (I(v_3) \cup I(v_7), I(v_6) \cup I(v_8)))$ is a fuzzy linear interval 2-join, in which $(I(v_3) \cup I(v_7), I(v_6) \cup I(v_8))$ is the only possible nonlinear homogeneous pair of cliques. So we can construct an $(l-1)$-colouring of $G - S$ by Lemma 7.1 as in the previous two proofs. This gives us an $l$-colouring of $S$.

So assume $I(v_1)$ is now empty but $I(v_4)$ is not. Clearly we can extend the $(l-1)$-colouring of $G_1 - S$ to a proper $(l-1)$-colouring of $(G_1 - S) \cup I(v_2)$. We claim that we now have an antihat 2-join and we can find an $(l-1)$-colouring of $G - S$ using Lemma 10.16.

The 2-join in $G - S$ is $((I(v_2), Y_1 \setminus S), ((I(v_3) \cup I(v_7)) \setminus S, Y_2 \setminus S))$. To see that $(G_2 - S) - (I(v_1) \cup I(v_2))$ is an antihat strip, we will relabel the vertices to conform with the definition of an antihat thickening. We relabel the sets $I(v_3)$, $I(v_{10})$, and $I(v_7)$ as $I(a_1)$, $I(a_2)$, and $I(a_3)$ respectively. We relabel $I(v_4)$ and $I(v_5)$ as $I(b_1)$ and $I(b_2)$ respectively. Finally, we relabel $I(v_6)$, $I(v_9)$, and $I(v_8)$ as $I(c_1)$, $I(c_2)$, and $I(c_3)$ (or $I(c_4)$ if $I(v_7) \cup I(v_8)$ is a clique) respectively. It is straightforward to confirm that this is an antihat strip. We therefore have an antihat 2-join in $G - S$, so by Lemma 10.16 we can find an $(l-1)$-colouring of $G - S$ and an $l$-colouring of $G$.

If $I(v_{10})$ is empty, then instead of taking vertices from $I(v_{10})$, $I(v_1)$ and $I(v_4)$, we take vertices from $I(v_1)$, $I(v_3)$ and $I(v_5)$, and proceed symmetrically. This time, we may worry that $I(v_3)$ will become empty, but in this case, since $I(v_{10})$ is also empty, we get a fuzzy linear interval 2-join exactly as in Case 1.

**Case 3:** $k = 0$; $l = |X_1| + |Y_1|$.

In this final case, every colour appears in $X_1 \cup Y_1$, and no colour appears twice. Therefore $X_2$ and $Y_2$ must receive colours appearing in $Y_1$ and $X_1$ respectively. Since $k$ is maximal, $l \geq |X_2| + |X_1| + \frac{1}{2}|Y_1|$ (from a vertex in $X_1$), and $l \geq |Y_2| + |Y_1| + \frac{1}{2}|X_1|$ (from a vertex in $Y_1$). It follows that $2l \geq \frac{3}{2}(|X_1| + |Y_1|) + |X_2| + |Y_2|$, so $|X_2| + |Y_2| \leq \frac{1}{2}l$.

Notice that $Z_2$ is cobipartite, and that the only non-edges in $Z_2$ are in $I(v_3) \cup I(v_6)$, $I(v_7) \cup I(v_8)$, and $I(v_9) \cup I(v_{10})$. We begin with an optimal colouring of $Z_2$, removing the colour classes of size two. Let $t_1$ be the number of such colour classes in $I(v_3) \cup I(v_6)$, and let $t$ be the total number of such colour classes. Denote these $2t$ vertices by $S$, noting that $Z_2 - S$ is a clique.

We construct an auxiliary graph $G'$ from $G_2 - S$ by adding all possible edges between $X_2$ and $Y_2$. Now $G'$ is cobipartite and perfect, and since a proper colouring of $G'$ will give vertices in $X_2$ and $Y_2$ distinct colours, it suffices to prove that $\omega(G') \leq l - t$. This gives us an $l$-colouring of $G_2$ in which no colour appears twice on $X_2 \cup Y_2$, so we can use it to extend the $l$-colouring of $G_1$ to an $l$-colouring of $G$.

Suppose there is a clique $W$ of size greater than $l - t$ in $G'$. We will now prove that $l - |X_2| - |Y_2| \geq \frac{1}{2}|Z_2| \geq t$, which implies that $W$ cannot be $X_2 \cup Y_2$. Consider vertices $u, v, x, y$ in $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$ respectively. Since every vertex in $Z_2$ has two neighbours in this set, the sum of the four degrees is at least $2(|X_1| + |X_2| + |Y_1| + |Y_2| + |Z_2|) - 4$. Therefore the sum $\gamma_l^j(u) + \gamma_l^j(v) + \gamma_l^j(x) + \gamma_l^j(y)$ is at least $4l \geq 2(|X_1| + |X_2| + |Y_1| + |Y_2|) + |Z_2|$. Thus $2l \geq |Z_2| + 2(|X_2| + |Y_2|)$, so $\frac{1}{2}|Z_2| + |X_2| + |Y_2| \leq l$.

A maximal clique $W$ in $G'$ intersecting both $I(v_1)$ and $I(v_2)$ as well as $Z_2$ must be $(I(v_1) \cup I(v_2) \cup I(v_7)) \setminus S$. But a vertex $v$ in $(I(v_7) \cap \Omega(v_7, v_8)) \setminus S$ (this set is nonempty because $(I(v_7), I(v_8))$ is a skeletal homogeneous pair) has either two neighbours or a twin in each stable set of size two in $S$. This means that if $|W| > l - t$, then $\gamma_l^j(v) > l$, a contradiction. So $W$ is not such a clique, and by symmetry $W$ does not intersect all three of $I(v_4)$, $I(v_5)$, and $I(v_8)$. A similar argument implies that $W$ cannot intersect only one of $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$. Since $|X_2| + |Y_2| \leq l - t$ we can see that $W$ cannot intersect three of these sets. Furthermore $|Z_2 - S| = \omega(Z_2) - t \leq l - t$, so $W$ cannot be contained in $Z_2 - S$. Therefore $W$ intersects all three of $X_2$, $Y_2$, and $Z_2$, and we can assume by symmetry that $W$ is $I(v_4) \setminus S$ and its neighbourhood in $X_2 \cup Y_2$, i.e. $(I(v_2) \cup I(v_3) \cup I(v_4)) \setminus S$.

Suppose that $|W| > l - t$. This inequality will provide us with new bounds on $l$, giving us a contradiction and completing the proof of the lemma. Let $u$ and $v$ be vertices in $I(v_2)$ and $I(v_4)$ respectively. Observe that $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3) \setminus S| + t$, since $u$ sees one vertex in every stable set in $S$. Thus $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3)| + (t - t_1)$, and likewise $d(v) + 1 \geq |Y_1| + |Y_2| + |I(v_3)| + (t - t_1)$. Since $I(v_2) \cup I(v_3) \cup I(v_{10}) \cup I(v_7)$ is a clique, it follows that $\omega'(u) \geq |I(v_2)| + |I(v_3)| + (t - t_1)$, because every stable set of $S$ hits $I(v_3) \cup I(v_7) \cup I(v_{10})$ exactly once. Likewise, $\omega'(v) \geq |I(v_4)| + |I(v_3)| + (t - t_1)$. The sum of these figures is at most $2\gamma_l^j(u) + 2\gamma_l^j(v)$, which is at most $4l$. This implies:

$$4l \geq (|X_1| + |Y_1|) + (|X_2| + |Y_2|) + 4(t - t_1) + 4|I(v_3)| + |I(v_2)| + I(v_4)|.$$

We know that $|X_1| + |Y_1| = l$, $|X_2| + |Y_2| > |I(v_2)| + |I(v_4)|$, and by assumption, $|I(v_2)| + |I(v_4)| + |I(v_3)| - t_1 > l - t$. Therefore,

$$\begin{aligned} 3l &\geq 2(|I(v_2)| + |I(v_4)| + |I(v_3)|) + 2|I(v_3)| + 4(t - t_1) \\ &\geq 2l + 2|I(v_3)| + 2(t - t_1) \end{aligned}$$

Thus $|I(v_3)| - t_1 \leq \frac{l}{2} - t$. And since $|X_2| + |Y_2| \leq \frac{l}{2}$, we get $|W| = |I(v_2)| + |I(v_3)| + |I(v_4)| - t_1 \leq l - t$, contrary to our assumption.

It follows that $\omega(G') \leq l - t$, so we can indeed complete the $l$-colouring of $G_2$ that is compatible with the colouring of $G_1$. This proves the lemma. □

Lemmas 10.16, 10.17, and 10.18 together immediately imply Lemma 10.15.

To deal with pseudo-line 2-joins we use another invariant, which is essentially a global analogue to the local $\gamma_l^j$. Define $\omega'(H_2)$ as $\max_{v \in H_2} \omega'(v)$, and define $\gamma_g^j(H_2)$ as $\lceil \frac{1}{2}(\Delta_G(H_2) + 1 + \omega'(H_2)) \rceil$. Note that $\Delta_G(H_2)$ is the maximum degree in $G$ over all vertices in $H_2$. Clearly $\gamma_l^j(H_2) \leq \gamma_g^j(H_2) \leq \gamma(G)$.

**Lemma 10.19.** *Suppose a skeletal claw-free graph $G$ admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join or a pseudo-line 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper $l$-colouring of $G_1$ for any $l \geq \gamma_g^j(H_2)$, we can find a proper $l$-colouring of $G$.*

*Proof.* We prove the lemma by induction on $l$. We let $G$ be a minimum counterexample, noting that $l = \gamma_g^j(H_2)$. Assume that $|X_1| \geq |Y_1|$.

If $((X_1, Y_1), (X_2, Y_2))$ is a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join, then the lemma is immediately implied by Lemma 10.15 given the observation that $\gamma_l^j(H_2) \leq \gamma_g^j(H_2)$. So we can assume that we have a pseudo-line 2-join.

Recall that $G_2$ is based on the line graph of a graph $J$, and the vertices of $J$ other than $j_1$, $j_2$, and $j_3$ are called *centres*. For a centre $t$ in $J$, we call the corresponding clique $C_t$. That is, $C_t = \cup I(e)$ over all vertices $e$ of $H$ whose corresponding edge in $J$ is incident to $t$. Let the edges $j_1j_2$ and $j_2j_3$ be $e_1$ and $e_2$ respectively. Note that $Z_2$ is a clique and so is $Z_2 \cup \Omega(e_1, e_2)$.

We begin by making the number $k$ of colours in $G_1$ that hit both $X_1$ and $Y_1$ maximal. First suppose that there is no colour class appearing in neither $X_1$ nor $Y_1$. As in the previous case, $l > |X_1|$. Since $k$ is maximal, there is a vertex $v \in X_1$ with a colour not appearing in $Y_1$, and it must have at least $l-1$ neighbours in $G_1$. This vertex is in $X_1 \cup X_2$, so $l = \gamma_g^j(H_2) \geq \frac{1}{2}l + \frac{1}{2}|X_1| + |X_2|$. Hence $l \geq |X_1| + 2|X_2|$. Since $l = |X_1| + |Y_1| - k$, we have $|X_2| \leq \frac{1}{2}|Y_1| - \frac{1}{2}k$. Now since $|X_2|$ is nonempty, $|Y_1| > k$ and there is a vertex in $Y_1$ with a colour not appearing in $X_1$. We can therefore apply the symmetric argument to prove that $l \geq \frac{1}{2}l + \frac{1}{2}|Y_1| + |Y_2|$, and consequently $|Y_2| \leq \frac{1}{2}|X_1| - \frac{1}{2}k$.

Observe that if $|Z_2| \leq \frac{1}{2}(|X_1| + |Y_1|)$ we can easily finish the colouring by giving $X_2$ colours appearing in $Y_1$ but not $X_1$, $Y_2$ colours appearing in $X_1$ but not $Y_1$, and $Z_2$ colours appearing in both $X_1$ and $Y_1$, and any leftover colours. In fact we can do this whenever $|Z_2| \leq l - |X_2| - |Y_2|$. So assume $|Z_2| > l - |X_2| - |Y_2|$. Let $A$ be a maximum clique in $G[X_2 \cup Z_2]$. Since $G[X_2 \cup Z_2]$ is cobipartite, we can colour it with $|A|$ colours, $|X_2|$ of which intersect $X_2$. Therefore if $|A| \leq l - |Y_2|$ we can colour $Y_2$ using colours that appear in $X_1$ but not in $Y_1$, then colour $X_2$ and $Z_2$ using $|A|$ colours such that those colours appearing in $X_2$ do not appear in $X_1$.

To see that $|A| \leq l - |Y_2|$, note that $\omega'(H_2) \geq |A|$ and since the degree of any vertex in $I(e_1)$ is at least $|X_1| + |X_2| + |Z_2| - 1$, $l = \gamma_g^j(H_2) \geq \frac{1}{2}(|A| + |Z_2| + |X_2| + |X_1|)$. Since $|Z_2| > l - |X_2| - |Y_2|$, this implies that $l > |A| + |X_1| - |Y_2| \geq |A| + \frac{1}{2}|Y_2|$. Therefore $|A| \leq l - |Y_2|$ and we can complete the $\gamma_g^j(G)$-colouring of $G$.

We can now assume that there is a colour class $S$ in $G_1$ that appears in neither $X_1$ nor $Y_1$. We will find a stable set $S_2$ in $G_2$ such that removing $S \cup S_2$ lowers $\gamma_g^j(H_2)$; this will imply that $\chi(G) \leq l$ by induction.

First note that if there are at most two centres then we actually have an antihat 2-join – this is straightforward to confirm as there are only five vertices in $J$. So we can assume that there are at least three centres.

Suppose we set $S_2$ to be a diad (i.e. a stable set of size two) in $G[I(e_1) \cup I(e_2)]$ such that $S_2$ intersects $\Omega(e_1, e_2)$ if it is nonempty. $S_2$ exists because $G[I(e_1) \cup I(e_2)]$ is not a clique. If removing $S \cup S_2$ does not lower $\omega_j(G)$, then there must be a maximal clique in $G_2$ disjoint from $S_2$. Such a clique must be $C_t$ for some centre $t$ that sees $j_1$, $j_2$, and $j_3$ in $J$.

The size of $C_t$ must be at least $\max\{|X_1 \cup X_2|, |Y_1 \cup Y_2|, |Z_2|\} > \frac{1}{3}|V(G_2)|$, so by the number of vertices in $G_2$ there can be at most two such "centre cliques" of size $\omega'(H_2)$, since they must be disjoint – call the other one $C_{t'}$ if it exists. If we let $S_2$ be a stable set corresponding to a matching in $J$ that hits three centres and in particular hits $t$ and (if it exists) $t'$, we can see that removing $S \cup S_2$ lowers $\omega_j(G)$ so we are done. This $S_2$ must exist because $C_t$ intersects all of $X_2$, $Y_2$, and $Z_2$, so we can find $S_2$ unless every other centre has neighbourhood $j_2$ in $J$. If this is the case we can again easily confirm that we have an antihat 2-join, so we are done. □

We can now prove Theorem 10.1:

*Proof of Theorem 10.1.* Let $G$ be a minimum counterexample. Then it is skeletal and contains no clique cutset. Furthermore, $G$ is not three-cliqued or an icosahedral thickening or an antihat thickening or a quasi-line graph. Therefore by Theorem 9.5, $G$ admits an antihat 2-join or a pseudo-line 2-join or a strange 2-join or a gear 2-join. Thus Lemma 10.19 contradicts the minimality of $G$, proving Theorem 10.1. □

## 10.6 Algorithmic Considerations

We now show that our proofs of Theorems 10.1 and 10.2 yield polynomial time algorithms for $\gamma(G)$- and $\gamma_l(G)$-colouring $G$, respectively.

By Theorem 6.11 we can assume that $G$ is skeletal. Furthermore we can identify sets of twin vertices in $G$ in polynomial time. This immediately implies that we can recognize skeletal antiprismatic thickenings and skeletal icosahedral thickenings in polynomial time. We deal with these two simple cases first.

If $G$ is an antiprismatic thickening, we can repeatedly remove triads from $G$, each time lowering $\gamma_l(G)$. These triads will be colour classes. What remains is a graph containing no triad; we can colour these optimally in polynomial time, since an optimal colouring corresponds to a maximum matching in the complement.

If $G$ is an icosahedral thickening, then observe that since $G$ is skeletal there are at most 12 equivalence classes of twin vertices; therefore there are at most $12^3$ different types of stable sets. We can formulate the problem of colouring $G$ as an integer program in which each variable represents the number of stable sets of a given type we use in the colouring. Each variable has size at most $n$, so we can exhaustively solve the problem in $O(n^{12^3})$ time. This is clearly not optimal.

We now consider the problem of colouring three-cliqued claw-free graphs.

### 10.6.1 Three-cliqued graphs

Maffray and Preissmann proved that it is *NP*-complete to decide whether or not a triangle-free graph is three-colourable [MP96]. Consequently it is *NP*-complete to decide whether or not a claw-free graph is three-cliqued. This makes dealing with three-cliqued claw-free graphs a slightly delicate issue. However, consider a claw-free graph $G$. If $\alpha(G) \leq 2$ we know we can optimally colour it in polynomial time. We will show that if $\alpha(G) = 3$, then in polynomial time we can either $\gamma_l(G)$-colour $G$, or determine that $G$ is not three-cliqued.

To prove that we can $\gamma_l(G)$-colour skeletal three-cliqued claw-free graphs in polynomial time, we first note that we can find a good triad efficiently:

**Lemma 10.20.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) \leq 3$. If $G$ contains a good triad then we can find one in polynomial time.*

*Proof.* To find $T$ we check every set of three vertices, of which there are $O(n^3)$, to see if it satisfies the requirements. Since we can efficiently determine which vertices are twins and which vertices trump which vertices, we can do this in polynomial time. □

**Lemma 10.21.** *Let $G$ be a skeletal claw-free graph with $\alpha(G) = 3$, and suppose $G$ contains no good triad. Then in polynomial time we can $\gamma_l(G)$-colour $G$ or determine that $G$ is not three-cliqued.*

*Proof.* We define the *triad graph* $t(G)$ of $G$. We let $V(t(G)) = V(G)$, and two vertices are adjacent in $t(G)$ precisely if some triad in $G$ contains both of them. We can easily find the components of $t(G)$ in polynomial time; there is at least one which is not a singleton.

Suppose that $G$ is three-cliqued. Then it admits a hex-join into terms $(G_1, A_1, B_1, C_1)$ and (possibly empty) $(G_2, A_2, B_2, C_2)$ such that $G_1$ is minimal and contains a triad. Since $G$ contains no good triad, it follows from the proofs of Lemmas 10.8, 10.9, 10.10, and 10.11 that $(G_1, A_1, B_1, C_1)$ is in $\mathcal{TTC}_1$. Furthermore the graph from which $G_1$ arises, i.e. $H$ such that $G_1 = L(H)$, has more than three centres and hence more than six vertices, otherwise $G$ would contain a good triad.

It is therefore easy to confirm that there is a component $X$ of $t(G)$ such that $X = V(G_1)$. We can find such a component $X$ in polynomial time, because any graph in $\mathcal{TTC}_1$ is a proper thickening of a line graph of a specific bipartite graph $H$. In particular we can find $(G_1, A_1, B_1, C_1)$ efficiently, because we can find $H$ efficiently and the definition of $\mathcal{TTC}_1$ implies that the choice of vertices $a$, $b$, and $c$ of $H$ is unique. Thus since $G_1$ is a term in a hex-join, we can determine $A_2$, $B_2$, and $C_2$ by taking a vertex in $G_2$ and looking at its neighbourhood in $G_1$, assuming that $G$ is three-cliqued.

We now proceed as in the proof of Lemma 10.12. With our base graph $H$ of $(G_1, A_1, B_1, C_1)$ in hand, it is not hard to see that we can decide which action is necessary in polynomial time. In each case we find a triad whose removal is guaranteed to lower $\gamma_l(G)$ or we remove edges from $G$ to reach a graph $G'$ such that $\chi(G') = \chi(G)$. From the proof of Lemma 10.12 it is clear that we can find $G'$ in polynomial time, and given a $k$-colouring of $G'$ we can find a $k$-colouring of $G$ in polynomial time. We can recursively $\gamma_l(G)$-colour $G'$ in polynomial time, possibly appealing to Lemma 10.20, since it is a proper subgraph of $G$.

Now suppose $G$ is not three-cliqued. If there is a component $X$ of $t(G)$ such that $G[X]$ is in $\mathcal{TTC}_1$, then again we have a unique choice of $a$, $b$, and $c$ in $H$ and a unique expression of $G[X]$ as $(G_1, A_1, B_1, C_1)$. Let $A_2$ be the set of vertices in $G - X$ with a neighbour in $A_1$ and a neighbour in $B_1$; we define $B_2$ and $C_2$ accordingly. Since $G$ is not three-cliqued, either $A_2$, $B_2$, and $C_2$ do not partition the vertices of $G - X$, or they are not all cliques. Either way we can determine this in polynomial time. □

Using these two lemmas we can prove the desired result:

**Theorem 10.22.** *Let $G$ be a claw-free graph with $\alpha(G) \geq 3$. Then in polynomial time we can either $\gamma_l(G)$-colour $G$ or determine that $\chi(\overline{G}) \geq 4$.*

*Proof.* By Theorem 6.11 we can assume $G$ is skeletal. If $G$ contains a good triad $T$, we can find $T$ in polynomial time and recursively $\gamma_l(G) - 1$ colour $G - T$. If $G$ does not contain a good triad, then the result follows immediately from Lemma 10.21. □

### 10.6.2 Graphs that are not three-cliqued

By Theorem 9.5 and Theorem 9.7, if $G$ is a skeletal claw-free graph that is not three-cliqued, then one of the following applies:

1. $G$ is an antiprismatic thickening
2. $G$ is an icosahedral thickening
3. $G$ is quasi-line
4. $G$ contains a clique cutset
5. $G$ admits an antihat 2-join or a pseudo-line 2-join or a strange 2-join or a gear 2-join.

We already know how to deal with all of these cases efficiently, either by colouring in polynomial time or reducing to a smaller colouring problem, except in the case of 2-joins. To see that we can find an antihat 2-join or a pseudo-line 2-join or a gear 2-join efficiently, recall Chapter 8, in which we describe the structure of such a 2-join relative to a $W_5$. Given the correct choice of a $W_5$ in $G$, we can easily find a $W_5$ 2-join or a gear 2-join $((X_1, Y_1), (X_2, Y_2))$ separating $G_1$ from $G_2$. There are $O(n^6)$ 5-wheels in $G$, so we can find such a 2-join in polynomial time.

Since $G$ is skeletal, we can easily check whether or not $G_2$ is a gear strip in polynomial time: a skeletal gear strip has at most twelve equivalence classes of twin vertices. So assume that we have an antihat 2-join or a pseudo-line 2-join or a strange 2-join. Checking if we have a strange 2-join is trivial. Checking if we have an antihat 2-join is straightforward once we determine the adjacency between $X_2$ and $Y_2$. Otherwise we have a pseudo-line 2-join. In this case, $I(e_1)$ and $I(e_2)$ are precisely those vertices in $X_2$ and $Y_2$ respectively that are complete to $G_2 - X_2 - Y_2$. Furthermore, adding all edges between $I(e_1)$ and $I(e_2)$ leaves us with a line graph, the structure of which we can easily determine. Thus we can find these desired 2-joins in polynomial time.

To reduce on these 2-joins, we now consider the proof of Lemmas 10.16, 10.17, 10.18, and 10.19. We do one of two things: reduce the size of the graph and apply induction, or complete the $l$-colouring of $G$ in one step. Just as with Lemma 7.1, showing that we can do this in polynomial time is straightforward given the proof of the lemma. Thus we get the desired algorithmic result:

**Theorem 10.23.** *For any claw-free graph $G$, we can $\gamma(G)$-colour $G$ in polynomial time.*

# Chapter 11

# Future Directions

We now conclude the thesis by describing some related questions that remain unanswered. They lie mainly in the two general areas of the Main Conjecture and claw-free graphs.

## Problems related to the Main Conjecture

The most prominent problem that remains open is Reed's conjecture on $\omega$, $\Delta$, and $\chi$, i.e. the Main Conjecture. An outright proof would represent an enormously significant result in the field of graph colouring. However, at this moment it seems quite far out of reach. Here we describe some more accessible steps towards a solution.

### Triangle-free graphs

The Main Conjecture is still open for triangle-free graphs, i.e. the case $\omega = 2$. In this case we get a special case of the Main Conjecture:

**Conjecture 11.1.** *For any triangle free graph $G$, $\chi(G) \leq \frac{1}{2}\Delta(G) + 2$.*

This conjecture is implied by Brooks' Theorem when $\Delta = 4$, and when $\Delta > 10^4$ it is implied by Johansson's result that triangle-free graphs have chromatic number at most $9\frac{\Delta}{\log \Delta}$ (see [MR00]). It is striking that the conjecture is open even for $\Delta \in \{5, 6, 7\}$, and although they are attractive cases to work on they should not be taken lightly. The case $\Delta = 6$ should be the easiest of the three, and would easily imply the case $\Delta = 5$. The triangle-free case of the Main Conjecture is of significant interest in its own right, and of further interest because it is a prerequisite to attacking the Main Conjecture for bull-free graphs using Chudnovsky's recent structure theorem [Chu08]. Since we know that the Main Conjecture holds for graphs containing no triad, this would give us the first nontrivial self-complementary class of graphs, other than perfect graphs, for which the Main Conjecture is known to be true (the class would be all graphs for which $\max\{\alpha, \omega\} = 2$).

Some steps towards the triangle-free case of the Main Conjecture have been made. Kostochka [Kos78] proved that the conjectured bound holds for graphs containing no cycle of length at most $4(\Delta + 2)\log \Delta$. And as mentioned in Chapter 2, he also proved that triangle free graphs have chromatic number at most $\frac{2\Delta}{3} + 2$.

Let $k_\Delta$ be the largest possible chromatic number of a triangle-free graph with maximum degree $\Delta$. Suppose we can prove that $k_5 \leq 4$, $k_7 \leq 5$, $k_{10} \leq 6$, $k_{13} \leq 7$, $k_{17} \leq 10$, and $k_{19} \leq 11$. Then we could combine these results using a classical result of Lovász to prove the Main Conjecture for all triangle-free graphs. Observe that the Main Conjecture requires all of these partial results except $k_{10} \leq 6$ and $k_{13} \leq 7$, which are stronger than the Main Conjecture's bound by one.

### Asymptotic approaches

Very little is known about asymptotic weakenings of the Main Conjecture. In the original paper, Reed made two weaker conjectures: $\chi \leq \frac{1}{2}(\Delta + \omega) + o(\Delta)$ and $\chi \leq \frac{1}{2}(\Delta + \omega) + o(\omega)$ [Ree98]. Both seem difficult.

If we insist that $\omega = o(\Delta)$, can we prove that $\chi \leq (\frac{1}{2} + o(1))\Delta$? With or without this restriction, can we prove that $\chi \leq (\frac{1}{2} + a)(\Delta + \omega)$ for some value of $a$ much smaller than $\frac{1}{2}$? The most obvious approach to this problem is the *Rödl Nibble*, in which groups of randomized stable sets are removed as colour classes and invariants related to $\Delta$ and $\omega$ are proven to decay at some desirable rate (see [MR00]). However, maintaining any kind of independence among randomly chosen stable sets (which hopefully allows us to apply the Lovász Local Lemma) poses a challenge when we need $\gamma$ to drop sufficiently between nibbles.

### Problems related to claw-free graphs

The work in this thesis has brought up several questions about fractional and integer colourings of claw-free graphs. We know that $\chi$ can be as high as $\frac{6}{5}\chi_f$ for claw-free graphs, and we know that $\chi$ and $\chi_f$ agree asymptotically for quasi-line graphs. When $\alpha \leq 2$ it is well-known that $\chi \leq \frac{6}{5}\chi_f$, and we think this extends to all claw-free graphs with stability number at most three. In contrast, we can generalize our approach to quasi-line graphs to show that the fractional and integer chromatic numbers agree asymptotically for claw-free graphs with $\alpha \geq 4$. The only remaining details concern the ten types of 1-joins defined by Chudnovsky and Seymour [CS08b]. These must be treated with some care because they force us to consider claw-free graphs with $\alpha(G) \leq 3$ containing a simplicial vertex. Our current case analysis is somewhat lacking in elegance – a deeper investigation into fractional colourings of three-cliqued claw-free graphs may yield a nicer solution.

Many other questions related to claw-free graphs remain. In Chapter 8 we gave a natural and easy reduction from claw-free graphs with $\alpha \geq 4$ to quasi-line graphs. However, at this time the only known proof of the structure theorems for quasi-line graphs are as a special case of claw-free graphs. A direct proof of the structure theorems would be a very nice result.

Finally, in Chapter 6 we introduced skeletal graphs, which have never been used before. Our work in this thesis tells us that homogeneous pairs of cliques should be a point of interest outside the area of perfect graph theory. Are there other classes of graphs that can be better described and manipulated using nonskeletal homogeneous pairs of cliques? Can skeletal homogeneous pairs be generalized in a useful way to homogeneous $k$-tuples of cliques? It would be very interesting to see whether or not skeletal graphs have useful applications outside the domain of this thesis.

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# Glossary

Here we use $G$ to denote a graph, $u$ and $v$ to denote vertices, $e$ to denote an dge, and $S$ and $T$ to denote vertex sets.

**Basic graph terminology**

| Term | Symbol | Meaning |
|---|---|---|
| graph | $G = (V, E)$ | a set $V$ of vertices and a set $E$ of unordered pairs of vertices |
| multigraph | $H = (V, E)$ | a set $V$ of vertices and a multiset $E$ of unordered pairs of vertices |
| vertex set of $G$ | $V(G)$ | the set of vertices of $G$ |
| edge set of $G$ | $E(G)$ | the set of edges of $G$ |
| clique | | a set of mutually adjacent vertices |
| stable set | | a set of mutually nonadjacent vertices |
| $u$ sees $v$ | | $u$ is adjacent to $v$<br>there is an edge between $u$ and $v$<br>$\{u, v\} \in E(G)$ |
| neighbour of $v$ | | a vertex adjacent to $v$ |
| neighbourhood of $v$ | $N(v)$ | the set of neighbours of $v$ |
| closed neighbourhood of $v$ | $\bar{N}(v)$ | $N(v) \cup \{v\}$ |
| matching | | a set of edges, no two of which share an endpoint |
| subgraph of $G$ induced on $S$ | $G[S]$ | $V(G[S]) = S$,<br>$E(G[S]) = \{uv \in E(G) \mid \{u, v\} \subseteq S\}$ |
| complement of $G$ | $\overline{G}$ | $V(\overline{G}) = V(G)$,<br>$E(\overline{G}) = \{uv \mid \{u, v\} \in V(G), uv \notin E(G)\}$ |

**Graph invariants**

| Term | Symbol | Meaning |
|---|---|---|
| clique number of $G$ | $\omega(G)$ | size of the largest clique in $G$ |
| stability number of $G$ | $\alpha(G)$ | size of the largest stable set in $G$ |
| degree of $v$ | $d(v)$ | size of $N(v)$ |
| maximum degree of $G$ | $\Delta(G)$ | $\max_{v\in V(G)} d(v)$ |
| minimum degree of $G$ | $\delta(G)$ | $\min_{v\in V(G)} d(v)$ |
| clique number of $G$ | $\omega(v)$ | $\omega(G[\tilde{N}(v)])$ |
| chromatic number of $G$ | $\chi(G)$ | |
| gamma of $G$ | $\gamma(G)$ | $\lceil\frac{1}{2}(\Delta(G)+1+\omega(G))\rceil$ |
| local gamma of $v$ | $\gamma_l(v)$ | $\lceil\frac{1}{2}(d(v)+1+\omega(v))\rceil$ |
| local gamma of $G$ | $\gamma_l(G)$ | $\max_{v\in V(G)} \gamma_l(v)$ |
| Main Conjecture | | for all $G$, $\chi(G)\le\gamma(G)$ |
| Local Strengthening | | for all $G$, $\chi(G)\le\gamma_l(G)$ |

**Further terminology**

| Term | Symbol | Meaning |
|---|---|---|
| claw | | a vertex $v$ and three mutually nonadjacent neighbours of $v$ |
| triad | | a stable set of size three |
| $S$ is complete to $T$ | | every possible edge between $S$ and $T$ exists |
| $S$ is anticomplete to $T$ | | no edges between $S$ and $T$ exists |
| hole | | an induced cycle of length $\geq 4$ in $G$ |
| antihole | | induced cycle of length $\geq 4$ in $\overline{G}$ |